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Dedicated to the 50th Anniversary of "Fuzzy Sets" Published by Lotfi A. Zadeh in 1965

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With Emphasis on the Integration of Three Technologies

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  - Computational Theory and Mathematics: Q3;
  - Computer Networks and Communications: Q2;
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Focus & Scope: International Journal of Computers Communications & Control is directed to the international communities of scientific researchers in computer and control from the universities, research units and industry.

To differentiate from other similar journals, the editorial policy of IJCCC encourages the submission of original scientific papers that focus on the integration of the 3 "C" (Computing, Communication, Control).

In particular the following topics are expected to be addressed by authors:

- Integrated solutions in computer-based control and communications;
- Computational intelligence methods (with particular emphasis on fuzzy logic-based methods, ANN, evolutionary computing, collective/swarm intelligence);
- Advanced decision support systems (with particular emphasis on the usage of combined solvers and/or web technologies).

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Lotfi A. Zadeh
(B. February 4, 1921)

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7th IFAC CONFERENCE on MANAGEMENT and CONTROL of PRODUCTION and LOGISTICS (MCPL 2016)

Bremen, Germany, February 22-24, 2016

http://www.mcpl2016.logdynamics.de

The 7th IFAC Conference on Management and Control of Production and Logistics (MCPL 2016) will be held in Bremen (Germany) from 22nd to 24th of February 2016 jointly with the 5th International Conference on Dynamics in Logistics (LDIC 2016) and several satellite events. Accepted papers will be published in the proceedings of the event using the open-access IFAC-PapersOnLine.

Scope of the Conference

The conference, sponsored by IFAC, aims to bring together researchers and practitioners from different areas of production and logistics with a special focus on the engineering side of management and control of such systems. The central idea is to establish a common ground in order to promote a synergy among different disciplines for exploring new solutions for complex scientific and technical challenges. The objectives of the conference are to provide high quality research and professional interactions for the advancement of science, technology and fellowship. It also provides the participants an opportunity to present their research papers and experience reports, and to take part in open discussions.

Topics

Topics of interest include, but are not limited to: *Modeling and Simulation; Decision-Support Systems: Concepts, Methods and Algorithms; Discrete Event Systems; Cyber-physical Production and Logistic Systems; Probabilistic and Statistical Modeling; Production Planning and Scheduling; Operational Research Applications; Control Methods and Concepts; Robotics and Man-Machine Interaction; Factory Automation; Intelligent Manufacturing Systems; Advanced Process Control and Wireless Automation; Lean Six Sigma: Enterprise, Manufacturing and Healthcare; ERP and Inventory Control; Management of Organizations; Supply Chain and Green Supply Chain Management; Urban Freight Distribution and City Logistics; Information Technology in Production, Logistics and Management; Humanitarian Logistics; Socio-technical and Cognitive Aspects in Manufacturing and Logistics; Quality Management Systems and Performance Indicators.*

MCPL

The IFAC MCPL 2016 is the 7th in a very successful series of events, previously held in Fortaleza (Brazil), Campinas (Brazil), Grenoble (France), Santiago (Chile), Sibiu (Romania) and Coimbra (Portugal). This seventh edition will be organized by the BIBA Bremer Institut für Produktion und Logistik, one of the most important research centers for Production and Logistic Systems in Europe. The conference will be held in the Hanseatic City Bremen on the banks of the River Weser, one of the biggest logistics hubs in Europe. The city combines high-tech and picturesque narrow streets built in centuries past, and its rich heritage of history is greatly cherished and lovingly preserved. Bremen is the only city in Germany to have an airport ten minutes away from the city. Destinations like London, Madrid, Munich or Vienna can be reached easily by plane. Submissions The conference submission tool "PaperCept" will open in July 2015. Papers submitted to the main conference must contain original research and should not exceed six pages. Simultaneous submission to other conferences with proceedings or submission of material that has already been published elsewhere is not allowed.

Program Chairs: Jurgen Pannek (pan@biba.uni-bremen.de)
and Florin Gheorghe Filip (ffilip@acad.ro).

6th INTERNATIONAL CONFERENCE on COMPUTERS, COMMUNICATIONS and CONTROL (ICCCC 2016)

Hotel President, Baile Felix, Oradea, Romania, May 10-14, 2016
Organized by Agora University of Oradea,
under the aegis of Romanian Academy: Information Science and Technology Section.
http://univagora.ro/en/icccc2016/

Scope and Topics

The International Conference on Computers Communications and Control (ICCCC) has been founded in 2006 by I. Dzitac, F.G. Filip and M.-J. Manolescu and organized every even year by Agora University of Oradea, under the aegis of the Information Science and Technology Section of Romanian Academy and IEEE - Romania Section.

The goal of this conference is to bring together international researchers, scientists in academia and industry to present and discuss in a friendly environment their latest research findings on a broad array of topics in computer networking and control.

The Program Committee is soliciting paper describing original, previously unpublished, completed research, not currently under review by another conference or journal, addressing state-of-the-art research and development in all areas related to computer networking and control.

In particular the following topics are expected to be addressed by authors:
1) Integrated solutions in computer-based control and communications;
2) Network Optimization and Security;
3) Computational intelligence methods (with particular emphasis on fuzzy logic-based methods, ANN, evolutionary computing, collective/swarm intelligence);
4) Data Mining and Intelligent Knowledge Management;
5) Advanced decision support systems (with particular emphasis on the usage of combined solvers and/or web technologies);
6) Membrane Computing - Theory and Applications;
7) Stereovision Based Perception for Autonomous Mobile Systems and Advanced Driving Assistance.

Special Sessions

Special Session 1: Network Optimization and Security, Organizer and Chair: Yezid DONOSO (Colombia);
Special Session 2: Data Mining and Intelligent Knowledge Management, Organizers and Chairs: Gang KOU (China) and Yi PENG (China);
Special Session 3: Computational Intelligence Methods, Organizers and Chairs: Razvan ANDONIE (USA) and Donald DAVENDRA (USA);
Special Session 4: Advanced Decision Support Systems, Organizer and Chair: Marius CIOCA (Romania) and Felisa CORDOVA (Chile);
Special Session 5: Fuzzy Control, Modeling and Optimization, Organizer and Chair: Radu-Emil PRECUP (Romania);
Special Session 6: Membrane Computing - Theory and Aplications, Organizers and Chairs: Marian GHEORGHE (UK) and Florentin IPATE (Romania);
Special Session 7: Stereovision Based Perception for Autonomous Mobile Systems and Advanced Driving Assistance, Organizer and Chair: Sergiu NEDEVSCHI (Romania).
Keynote Speakers: Enrique HERRA VIEDMA(Spain), Zenonas TURSKIS (Lithuania), Gang KOU (China).
Conference Chairs: Ioan DZITAC, Florin Gheorghe FILIP and Misu-Jan MANOLESCU.

International Journal of Computers, Communications & Control
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Contents

INTERNATIONAL JOURNAL OF COMPUTERS COMMUNICATIONS & CONTROL
Special Issue on Fuzzy Sets and Applications (Celebration of the 50th Anniversary of Fuzzy Sets)
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Foreword

R.R. Yager

Ronald R. Yager[1]
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Here we are celebrating the fiftieth anniversary of Lotfi Zadeh's pioneering paper Fuzzy Sets that appeared in Information and Control in 1965. While the paper was clear, direct and easy to understand the ideas presented were revolutionary and ground breaking. This article now has close to sixty thousand citations as noted in Google Scholar. Clarity and simplicity are the hallmark of the writing of Zadeh. This has always reminded me of the writings of Sigmund Freud. One rarely needs to draw on complex mathematics to read Zadeh's papers.

The capacity of fuzzy sets to represent and manage imprecise linguistic concepts has proven to be of great use in the modern technological world where there is now a great interest in building intelligent systems that can model human reasoning but take advantage of the vast amount of information available on the Internet. If the idea of fuzzy sets was not introduced in Zadeh's ground breaking paper in the 1960's it would have naturally arisen in early 2000's as we moved into intelligent systems. However the early reception of fuzzy sets was not very promising both the Artificial Intelligence community and the probabilistic community were very dubious of the worth of this new field. Interestingly a number of researchers from Romania, the home of this journal, were among the early supporters of ideas presented by Zadeh. Zadeh persevered in the face of adversity, describing himself as thick skinned, until mid 1980's when the Japanese engineers provided significant applications of fuzzy sets in control systems. Particularly notable among these applications was the use of fuzzy control to the Sendai subway. These applications brought a new appreciation to the possibilities of fuzzy sets and clearly changed its history.

The editors of this Special Issue of the International Journal of Computers Communications & Control dedicated to the 50th anniversary of the publication of Lotfi Zadeh's pioneering paper Fuzzy Sets have provided a collection of papers representative of the current state of the field of fuzzy sets. Included in this issue are papers investigating some current theoretical issues and applied papers in domains in which fuzzy sets has introduced some benefits.

The editorial team is to be congratulated for providing a wonderful anniversary gift to Professor Zadeh and a useful collection of articles for the community.

Ronald R. Yager
New York, October 2015

[1]Editor's note: "Ronald R. Yager is Professor of Information Systems and Director of the Machine Intelligence Institute at Iona College. He is among the world's most highly cited researchers with over 47,000 citations (updated by editor) to his work in Google Scholar. He is editor and chief of the International Journal of Intelligent Systems and serves on the editorial board of numerous journals. He has published over 500 papers and edited over 30 books in areas related to fuzzy sets, human behavioral modeling, decision-making under uncertainty and the fusion of information. He was the recipient of the IEEE Computational Intelligence Society Pioneer award in Fuzzy Systems. He received the special honorary medal of the 50-th Anniversary of the Polish Academy of Sciences. He received the Lifetime Outstanding Achievement Award from International the Fuzzy Systems Association. He received honorary doctorate degrees, honoris causa, from the Azerbaijan Technical University and the State University of Information Technologies, Sofia Bulgaria. Dr. Yager is a fellow of the IEEE, the New York Academy of Sciences and the Fuzzy Systems Association. He has served at the National Science Foundation as program director in the Information Sciences program. He was a NASA/Stanford visiting fellow and a research associate at the University of California, Berkeley. He has been a lecturer at NATO Advanced Study Institutes. He received his undergraduate degree from the City College of New York and his Ph. D. from the Polytechnic Institute New York University." Source: http://www.iona.edu retrieved on 2th October 2015.

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The Fuzzification of Classical Structures: A General View

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Abstract: The aim of this survey article, dedicated to the 50th anniversary of Zadeh's pioneering paper "Fuzzy Sets" (1965), is to offer a unitary view to some important spaces in fuzzy mathematics: fuzzy real line, fuzzy topological spaces, fuzzy metric spaces, fuzzy topological vector spaces, fuzzy normed linear spaces. We believe that this paper will be a support for future research in this field.
Keywords: Fuzzy real line, fuzzy topological spaces, fuzzy metric spaces, fuzzy topological vector spaces, fuzzy normed linear spaces, fuzzy F-space.

1 Introduction

An introduction in the classical set theory begins, in general, in the following way: by a set we understand a collection of objects, well individualized, such that we can decide without any ambiguity whether a given element belongs to that set or not. What should we do when we cannot answer this question? Can we talk about sets described in natural language such as "the set of beautiful women" or "the set of tall men"? Although these questions are natural they were formulated only in 1965 by Lotfi A. Zadeh. In order to give answers to these questions, L.A. Zadeh [61] introduced the concept of fuzzy set.

We present bellow some thoughts of Lotfi A. Zadeh, remembering the beginnings and the current impact of fuzzy sets theory.

In [63] Lotfi A. Zadeh said: "In July of 1964, I was attending a conference in New York and was staying at the home of my parents. They were away. I had a dinner engagement but it had to be canceled. I was alone in the apartment. My thoughts turned to the unsharpness of class boundaries. It was at that point that the simple concept of a fuzzy set occurred to me. It did not take me long to put my thoughts together and write a paper on the subject. This was the genesis of fuzzy set theory. I knew that the word "fuzzy" would make the theory controversial. Knowing how the real world functions, I submitted my paper to Information and Control because I was a member of the Editorial Board. There was just one review-which was very lukewarm. I believe that my paper would have been rejected if I were not on the Editorial Board. Today (20 Dec. 2010), with over 26,000 Google Scholar citations, "Fuzzy Sets"is by far the highest cited paper in Information and Control.

My paper was a turning point in my research. Since 1965, almost all of my papers relate to fuzzy set theory and fuzzy logic. As I expected, my 1965 paper drew a mixed reaction, partly because the word "fuzzy" is generally used in a pejorative sense, but, more substantively, because unsharpness of class boundaries was not considered in science and engineering. In large measure, comments of my paper were skeptical or hostile. An exception was Japan. In 1968, I began to receive letters from Japan expressing interest in application of fuzzy set theory to pattern recognition. In the years which followed, in Japan fuzzy set theory and fuzzy logic became objects of extensive research and wide-ranging application, especially in the realm of consumer products. A very visible application was the subway system in the city of Sendai - a fuzzy logic-based system designed by Hitachi and Kawasaki Heavy Industry. The system began to operate in 1987 and is considered to be a great success."

On October 2, 2015 the paper "Fuzzy Sets" has already over 58,540 citations in Google Scholar and all Zadeh's papers have over 151,300 citations.

"Computation with information described in natural language (NL) is closely related to Computing with Words. NL-Computation is of intrinsic importance because much of human knowledge is described in natural language. This is particularly true in such fields as economics, data mining, systems engineering, risk assessment and emergency management. It is safe to predict that as we move further into the age of machine intelligence and mechanized decision-making, NL-Computation will grow in visibility and importance." (L.A. Zadeh, [65]).

"What is thought-provoking is that neither traditional mathematics nor standard probability theory has the capability to deal with computational problems which are stated in a natural language. Not having this capability, it is traditional to dismiss such problems as ill-posed. In this perspective, perhaps the most remarkable contribution of Computing with Words (CW) is that it opens the door to empowering of mathematics with a fascinating capability - the capability to construct mathematical solutions of computational problems which are stated in a natural language. The basic importance of this capability derives from the fact that much of human knowledge, and especially world knowledge, is described in natural language. In conclusion, only recently did I begin to realize that the formalism of CW suggests a new and challenging direction in mathematics - mathematical solution of computational problems which are stated in a natural language. For mathematics, this is an unexplored territory." (L.A. Zadeh, [64]).

Since then many authors have developed the theory of fuzzy set and its applications. Especially, many mathematicians tried to extend in fuzzy context classical mathematics results. The success of the research undertaken has been demonstrated in a variety of areas such as: artificial intelligence, computer science, quantum particle physics, control engineering, robotics and many more. Perhaps the main reason for this rapid development is that the world that surrounds us is full of uncertainty, the data we collect from the environment are, in general, vague and incorrect. So the notion of fuzzy set allows us to study the degree of uncertainty mentioned above in a purely mathematical way.

2 Fuzzy Sets

The concept of fuzzy set was introduced by L.A. Zadeh [61] in 1965.

Definition 1. [61] A fuzzy set in X is a function $\mu : X \to [0,1]$. We denote by $\mathcal{F}(X)$ the family of all fuzzy sets in X.

Remark 2. In fact μ is the membership function of a fuzzy set A of X and the value $\mu(x)$ represents "the grade of membership" of x to fuzzy set A. But, in this paper, we adopt the convention to identify fuzzy sets with their membership functions. This identification was first used by J.A. Goguen [19].

Remark 3. As any subset of X can be identified with its characteristic function we remark that fuzzy sets generalize subsets.

Definition 4. [61] Let μ, ν be fuzzy sets in X. The union of fuzzy sets μ şi ν, denoted $\mu \vee \nu$, the intersection of fuzzy sets μ şi ν, denoted $\mu \wedge \nu$, the complement of fuzzy set μ, denoted $1 - \mu$, are fuzzy sets in X, defined by

$$(\mu \vee \nu)(x) = \max\{\mu(x), \nu(x)\} \tag{1}$$
$$(\mu \wedge \nu)(x) = \min\{\mu(x), \nu(x)\} \tag{2}$$
$$\mathcal{C}(\mu)(x) = 1 - \mu(x) \tag{3}$$

Definition 5. The union of the fuzzy sets $\{\mu_i\}_{i\in I}$ is defined by

$$\left(\bigvee_{i\in I} \mu_i\right)(x) = \sup\{\mu_i(x) \ : \ i \in I\} .$$

The intersection of the fuzzy sets $\{\mu_i\}_{i\in I}$ is defined by

$$\left(\bigwedge_{i\in I} \mu_i\right)(x) = \inf\{\mu_i(x) \ : \ i \in I\} .$$

Definition 6. Let $\alpha \in (0, 1]$, and let μ be a fuzzy set in X. The α-level set $[\mu]_\alpha$ is defined by

$$[\mu]_\alpha := \{x \in X \ : \ \mu(x) \geq \alpha\} .$$

The support of μ is

$$\mathrm{supp}\, \mu := \{x \in X \ : \ \mu(x) > 0\} .$$

Definition 7. [61] Let X be a vector space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$). A fuzzy set μ is called convex if

$$\mu(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu(x_1), \mu(x_2)\} , \ (\forall)x_1, x_2 \in X, (\forall)\lambda \in [0, 1] .$$

The extension principle is undoubtedly one of the most important of Zadeh's contribution in fuzzy set theory, allowing to extend in a fuzzy context almost any mathematical concept. The extension principle was introduced by Zadeh [61] in 1965, and then it suffered many changes: Zadeh [62]; Jain [24]; Dubois & Prade [14]. For more details of this principle and its extensions we refer the reader to [66], [30].

Let $X = X_1 \times X_2 \times \cdots \times X_r$ and $\mu_1, \mu_2, \cdots, \mu_r$ be fuzzy sets in $X_1, X_2, \cdots, X_r$, respectively. Let $f : X \to Y$. The extension principle allows us to define a fuzzy set in Y by

$$\mu(y) = \begin{cases} \sup\limits_{(x_1,\cdots,x_r)\in f^{-1}(y)} \min\{\mu_1(x_1), \cdots, \mu_r(x_r)\} & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{if} \quad f^{-1}(y) = \emptyset \end{cases} .$$

3 Fuzzy relations

It is well known that the fuzzy relations play an important role in fuzzy modeling and fuzzy control and they also have important applications in relational databases, approximate reasoning, preference modeling, medical diagnosis.

The concept of fuzzy relation was introduced by L.A. Zadeh in his classical paper [61]. According to L.A. Zadeh a fuzzy relation T between two nonempty sets X and Y is a fuzzy set in $X \times Y$, i.e. it is a mapping $T : X \times Y \to [0, 1]$. We denote by $FR(X, Y)$ the family of all fuzzy relations between X and Y. For $x \in X$ we denote by T_x the fuzzy set in Y defined by $T_x(y) = T(x, y)$. Thus, a fuzzy relation can be seen as a mapping $X \ni x \mapsto T_x \in \mathcal{F}(Y)$, where $\mathcal{F}(Y)$ represents the family of all fuzzy sets in Y.

Such mappings were investigated by various mathematicians under different aspects. Thus N. Papageorgiou [46] called these mappings fuzzy multifunctions and studied the continuity of these mappings. E. Tsiporkova, B. De Baets, E. Kerre [56, 57] called these maps fuzzy multivalued mappings and they defined lower and upper semi-continuous fuzzy multivalued mapping. The relationships between these two types were studied completely. The continuity of fuzzy multifunctions was also studied by I. Beg [6]. In papers [7, 8], I.Beg studied the linear fuzzy multivalued operators and vector-valued fuzzy multifunctions. An application $T : \mathbb{R}^m \to \mathcal{F}(\mathbb{R}^n)$ is called a fuzzy process (see Y. Chalco-Cano, M.A. Rojas-Medar, R. Osuna-Gómez [9]).

A special attention was given to convex fuzzy processes. They were introduced by M. Matloka [36] in 2000. Another concept of convex fuzzy process was proposed by Y. Syau, C. Low and T. Wu [55] in 2002. A comparative study of these fuzzy convex processes was made in 2010 by D. Qiu, F. Yang, L. Shu [47]. To avoid any confusion D. Qiu, F. Yang and L. Shu called the former M-convex fuzzy process and the latter SLW-convex process.

In paper [41] special types of fuzzy relations on vector spaces were considered : affine fuzzy relations, linear fuzzy relations, convex fuzzy relations, M-convex fuzzy relations. Some fundamental properties of fuzzy linear relations between vector spaces are considered in [43].

The domain $D(T)$ of T is a fuzzy set in X defined by $D(T)(x) := \sup\limits_{y\in Y} T(x, y)$ (see [56]). We note that

$$\text{supp } D(T) = \{x \in X \ : \ T_x \neq \emptyset\} = \{x \in X \ : \ (\exists) y \in Y \text{ such that } T(x, y) > 0\} .$$

If for all $x \in \text{supp } D(T)$ there exists unique $y \in Y$ such that $T(x, y) > 0$, then T is called fuzzy function (or single-valued fuzzy function). In this case, we denote this unique y by $T(x)$.

If $\mu \in \mathcal{F}(X)$, then $T(\mu) \in \mathcal{F}(Y)$ is defined by $T(\mu)(y) := \sup\limits_{x\in X}[T(x, y) \wedge \mu(x)]$ (see [6]). In particular, the range $R(T)$ of T is a fuzzy set in Y defined by $R(T)(y) := \sup\limits_{x\in X} T(x, y)$ [56].

Let $T \in FR(X, Y), S \in FR(Y, Z)$. The composition $S \circ T \in FR(X, Z)$ (or simply ST) is defined by $(S \circ T)(x, z) := \sup\limits_{y\in Y}[T(x, y) \wedge S(y, z)]$ [61].

Proposition 8. Let $T \in FR(X, Y), S \in FR(Y, Z)$. Then $(S \circ T)_x = S(T_x), (\forall)x \in X$.

Proposition 9. The operation " $\circ$ " is associative.

The inverse (or reverse relation) T^{-1} of a fuzzy relation $T \in FR(X, Y)$ is a fuzzy set in $Y \times X$ defined by $T^{-1}(y, x) = T(x, y)$. It is obvious that $R(T) = D(T^{-1})$ and $R(T^{-1}) = D(T)$. We remark that, for $\mu \in \mathcal{F}(Y)$, we have $T^{-1}(\mu)(x) = \sup\limits_{y\in Y}[T^{-1}(y, x) \wedge \mu(x)] = \sup\limits_{y\in Y}[T(x, y) \wedge \mu(x)]$. This type of inverse is usually called lower inverse [6].

4 Fuzzy real numbers

For the concept of fuzzy real number, arithmetic operation and ordering on the set of all fuzzy real numbers we refer the reader to the papers [13, 14, 17, 25, 26, 38, 59].

Definition 10. A fuzzy set in $\mathbb{R}$, namely a mapping $x : \mathbb{R} \to [0, 1]$, with the following properties:

1. x is convex, i.e. $x(t) \geq \min\{x(s), x(r)\}$, for $s \leq t \leq r$;

2. x is normal, i.e. $(\exists)t_0 \in \mathbb{R} \ : \ x(t_0) = 1$;

3. x is upper semicontinuous, i.e.

$$(\forall)t \in \mathbb{R}, (\forall)\alpha \in (0, 1] \ : \ x(t) < \alpha,$$
$$(\exists)\delta > 0 \text{ such that } |s - t| < \delta \Rightarrow x(s) < \alpha$$

is called a fuzzy real number. We will denote by $\mathbb{R}(I)$ the set of all fuzzy real numbers.

Remark 11. Let $x \in \mathbb{R}(I)$. For all $\alpha \in (0,1]$, the α-level sets $[x]_\alpha = \{t : x(t) \geq \alpha\}$ are closed intervals $[a_\alpha, b_\alpha]$, where the values $a_\alpha = -\infty$ and $b_\alpha = \infty$ are admissible. When $a_\alpha = -\infty$, the interval $[a_\alpha, b_\alpha]$ will be denoted by $(-\infty, b_\alpha]$.

Definition 12. A fuzzy real number x is called non-negative if $x(t) = 0, (\forall)t < 0$. The set of all non-negative real numbers will be denoted by $\mathbb{R}^*(I)$.

Remark 13. For each $r \in \mathbb{R}$ we can consider the fuzzy real number $\overline{r}$ defined by

$$\overline{r}(t) = \begin{cases} 1 & \text{if} \quad t = r \\ 0 & \text{if} \quad t \neq r \end{cases} .$$

These fuzzy numbers are called crisp. Thus $\mathbb{R}$ can be embedded in $\mathbb{R}(I)$.

Definition 14. [38] The arithmetic operations $+, -, \cdot, /$ on $\mathbb{R}(I)$, are defined by:

$$(x+y)(t) = \bigvee_{s \in \mathbb{R}} \min\{x(s), y(t-s)\}, \ (\forall)t \in \mathbb{R} \tag{4}$$

$$(x-y)(t) = \bigvee_{s \in \mathbb{R}} \min\{x(s), y(s-t)\}, \ (\forall)t \in \mathbb{R} \tag{5}$$

$$(xy)(t) = \bigvee_{s \in \mathbb{R}^*} \min\{x(s), y(t/s)\}, \ (\forall)t \in \mathbb{R} \tag{6}$$

$$(x/y)(t) = \bigvee_{s \in \mathbb{R}} \min\{x(ts), y(s)\}, \ (\forall)t \in \mathbb{R} \tag{7}$$

Remark 15. Previous definitions are special cases of Zadeh's extension principle.

Remark 16. The additive and multiplicative operations are associative and commutative with the identities $\overline{0}$ and $\overline{1}$, where

$$\overline{0}(t) = \begin{cases} 1 & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t \neq 0 \end{cases} , \ \overline{1}(t) = \begin{cases} 1 & \text{if} \quad t = 1 \\ 0 & \text{if} \quad t \neq 1 \end{cases} .$$

Remark 17. It is obvious that

1. $-x = \overline{0} - x$;
2. $(-x)(t) = x(-t)$;
3. $x - y = x + (-y)$;
4. $-(x+y) = (-x) + (-y)$.

Definition 18. The absolute value $|x|$ of $x \in \mathbb{R}(I)$ is defined by

$$|x|(t) = \begin{cases} \max\{x(t), x(-t)\} & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases} .$$

Proposition 19. [26] The equations $a + x = \overline{0}$ and $ax = \overline{1}$ have unique solutions if and only if a is crisp.

Definition 20. [16] A partial ordering on $\mathbb{R}(I)$ is defined by

$$x \leq y \text{ if } a^1_\alpha \leq a^2_\alpha \text{ and } b^1_\alpha \leq b^2_\alpha \ , \ (\forall)\alpha \in (0,1] \ ,$$

where $[x]_\alpha = [a^1_\alpha, b^1_\alpha]$ and $[y]_\alpha = [a^2_\alpha, b^2_\alpha]$.

Proposition 21. [26] If $[a_\alpha, b_\alpha]$, $0 < \alpha \leq 1$, are the α-level sets of a fuzzy real number x, then:

1. $[a_{\alpha_1}, b_{\alpha_1}] \supseteq [a_{\alpha_2}, b_{\alpha_2}]$, $(\forall) 0 < \alpha_1 \leq \alpha_2$;
2. $[\lim\limits_{k\to\infty} a_{\alpha_k}, \lim\limits_{k\to\infty} b_{\alpha_k}] = [a_\alpha, b_\alpha]$, where $\{\alpha_k\}$ is an increasing sequence in $(0,1]$ converging to α.

Conversely, if $[a_\alpha, b_\alpha]$, $0 < \alpha \leq 1$, is a family of non-empty intervals which satisfy the conditions (1) and (2), then the family $[a_\alpha, b_\alpha]$ represents the α-level sets of a fuzzy real number.

Remark 22. As α-level sets of a fuzzy real number is an interval, there is a debate in the nomenclature of fuzzy real numbers. In [15], D. Dubois and H. Prade suggested to call this fuzzy interval. They developed a different notion of fuzzy real number by considering it as a fuzzy element of the real line.

5 Fuzzy topological spaces

From the notion of fuzzy set, to the notion of fuzzy topological space, there was one more step to be taken. Thus, in 1968, C.L. Chang [10] introduced the notion of fuzzy topological space. The definition is a natural translation to fuzzy sets of the ordinary definition of topological space. Indeed, a fuzzy topology is a family $\mathcal{T}$, of fuzzy sets in X, such that $\mathcal{T}$ is closed with respect to arbitrary union and finite intersection and $X, \emptyset \in \mathcal{T}$.

Definition 23. [10] Let X be an arbitrary set. A fuzzy topology on X is a family $\mathcal{T} \subset \mathcal{F}(X)$ satisfying the following axioms:

1. $\emptyset, X \in \mathcal{T}$, where $\emptyset$ is characterized by the membership function $\mu(x) = 0, (\forall) x \in X$ and X is characterized by the membership function $\mu(x) = 1, (\forall) x \in X$;
2. If $\mu_1, \mu_2 \in \mathcal{T}$, then $\mu_1 \wedge \mu_2 \in \mathcal{T}$;
3. If $\{\mu_i\}_{i\in I} \subset \mathcal{T}$, then $\bigvee\limits_{i\in I} \mu_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ will be called fuzzy topological space. The elements of $\mathcal{T}$ will be called open fuzzy sets.

Definition 24. [10] Let $(X, \mathcal{T})$ be a fuzzy topological space. A fuzzy set μ_1 is a neighborhood of a fuzzy set μ_2 if there exists an open fuzzy set μ such that $\mu_2 \subseteq \mu \subseteq \mu_1$.

Theorem 5.1. Let $(X, \mathcal{T})$ be a fuzzy topological space. A fuzzy set μ is an open fuzzy set if and only for each fuzzy set $\mu_2 \subseteq \mu$, we have that μ is a neighborhood of μ_2.

Definition 25. [10] Let X, Y be arbitrary sets and $f : X \to Y$. If μ is a fuzzy set in Y, then the inverse of μ, denoted as $f^{-1}(\mu)$, is a fuzzy set in X defined by

$$f^{-1}(\mu)(x) := \mu(f(x)), (\forall) x \in X \ .$$

Conversely, if μ is a fuzzy set in X, the image of μ, denoted as $f(\mu)$, is a fuzzy sets in Y defined by

$$f(\mu)(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases} .$$

Definition 26. A function f from a fuzzy topological space $(X, \mathcal{T})$ to a fuzzy topological space $(Y, \mathcal{G})$ is said to be fuzzy continuous if the inverse of each open fuzzy set is an open fuzzy set.

In 1976, R. Lowen [33] remarked that with Chang's definition constant functions between fuzzy topological spaces are not necessarily continuous. Thus R. Lowen suggested an alternative and more natural definition replacing the condition $X, \emptyset \in \mathcal{T}$ with every constant function belong to $\mathcal{T}$.

Let $(X, \mathcal{T})$ be a topological space. We recall that a function $f : X \to \mathbb{R}$ is said to be lower semi-continuous if for all $a \in \mathbb{R}$, $\{x \in X \ : \ f(x) > a\}$ is an open set in X.

Example 27. [33] Let $(X, \mathcal{T})$ be a topological space. The lower semi-continuous fuzzy topology on X associated with $\mathcal{T}$ is

$$\omega(\mathcal{T}) := \{\mu : X \to [0,1] \ : \ \mu \text{ is lower semi-continuous}\} \ .$$

The usual fuzzy topology on $\mathbb{K}$ is the lower semi-continuous fuzzy topology generated by the usual topology of $\mathbb{K}$.

Remark 28. [34] If $(X, \mathcal{T}_i)_{i\in I}$ is a family of topological spaces and $\mathcal{T}$ is the product topology on $X = \prod\limits_{i\in I} X_i$, then $\omega(\mathcal{T})$ is the product of fuzzy topologies $\omega(\mathcal{T}_i), i \in I$.

Definition 29. [33] The closure and the interior of a fuzzy set μ in a fuzzy topological space $(X, \mathcal{T})$ are defined by

$$\overline{\mu} = \inf\{\mu_1 \ : \ \mu \subseteq \mu_1 \text{ and } \mathcal{C}(\mu_1) \in \mathcal{T}\}$$

$$\overset{\circ}{\mu} = \sup\{\mu_1 \ : \ \mu_1 \subseteq \mu \text{ and } \mu_1 \in \mathcal{T}\} \ .$$

We must note that, in paper [37], J. Michálek defined and studied another concept of fuzzy topological space which is quite different from the classic Chang's definition. In paper [35] it is shown the divergences between these two types of fuzzy topological spaces.

In paper [58], it is shown that the fuzzy continuous functions can be characterized by the closure of fuzzy sets, a subbasis of a fuzzy topology, and a fuzzy neighborhood.

In [53] a more consistent approach to the use of ideas of fuzzy mathematics in general topology has been developed.

Definition 30. [53] A fuzzy topological space is a pair $(X, \mathcal{T})$, where X is an arbitrary set and $\mathcal{T} : \mathcal{F}(X) \to [0,1]$ is a map satisfying the following axioms:

1. $\mathcal{T}(0) = \mathcal{T}(1) = 1$;
2. $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2), (\forall)\mu_1, \mu_2 \in \mathcal{F}(X)$;
3. $\mathcal{T}\left(\bigvee\limits_{i\in I} \mu_i\right) \geq \bigwedge\limits_{i\in I} \mathcal{T}(\mu_i), (\forall)\{\mu_i\}_{i\in I} \subseteq \mathcal{F}(X)$.

A nice survey concerning fuzzy topological spaces was written by A.P. Shostak [54]. This survey contains: various approaches to the definition of fuzzy topology, fundamental interrelations between the categories of fuzzy topology and the category of topological spaces, the notion of a fuzzy point, the convergence structure in fuzzy spaces, important topological properties for fuzzy spaces etc.

6 Fuzzy metric spaces

One of the important problems concerning the fuzzy topological spaces is to obtain an adequate notion of fuzzy metric space. Many authors have investigated this question, and several notions of fuzzy metric space have been defined and studied. We mention that the concept of fuzzy metric was introduced by I. Kramosil and J. Michálek [9] in 1975. Their notion is equivalent, in certain sense, with that of statistical metric. We note that the statistical metrics were studied many years before, and a brief survey on them was made by B. Schweizer and A. Sklar in paper [52]. We also note that, in 1994, A. George and P. Veeramani [18] modified the definition of fuzzy metric in order to obtain a Hausdorff topology on a fuzzy metric space.

Definition 31. [52] A binary operation

$$* : [0,1] \times [0,1] \to [0,1]$$

is called triangular norm (t-norm) if it satisfies the following condition:

1. $a * b = b * a, (\forall) a, b \in [0,1]$;
2. $a * 1 = a, (\forall) a \in [0,1]$;
3. $(a * b) * c = a * (b * c), (\forall) a, b, c \in [0,1]$;
4. If $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$, then $a * b \leq c * d$.

Example 32. Three basic examples of continuous t-norms are $\wedge, \cdot, *_L$, which are defined by $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ (usual multiplication in $[0,1]$) and $a *_L b = \max\{a + b - 1, 0\}$ (the Lukasiewicz t-norm).

Definition 33. [9] The triple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy metric, i.e. a fuzzy set in $X \times X \times [0, \infty)$ which satisfies the following conditions:

(M1) $M(x, y, 0) = 0, (\forall) x, y \in X$;

(M2) $[M(x, y, t) = 1, (\forall) t > 0]$ if and only if $x = y$;

(M3) $M(x, y, t) = M(y, x, t), (\forall) x, y \in X, (\forall) t \geq 0$;

(M4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s), (\forall) x, y, z \in X, (\forall) t, s \geq 0$;

(M5) $(\forall) x, y \in X$, $M(x, y, \cdot) : [0, \infty) \to [0,1]$ is left continuous and $\lim_{t \to \infty} M(x, y, t) = 1$

Remark 34. In the definition of the fuzzy metric space, I. Kramosil and J. Michálek have imposed another condition: "$M(x, y, \cdot)$ is nondecreasing, for all $x, y \in X$". M. Grabiec [12] showed that this statement derives from the other axioms.

Indeed, for $0 < t < s$, we have

$$M(x, y, s) \geq M(x, x, s - t) * M(x, y, t) = 1 * M(x, y, t) = M(x, y, t).$$

Example 35. [18] Let (X, d) be a metric space. Let

$$M_d : X \times X \times [0, \infty), M_d(x, y, t) = \begin{cases} \frac{t}{t + d(x,y)} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t = 0 \end{cases}.$$

Then $(X, M_d, \wedge)$ is a fuzzy metric space. M_d is called standard fuzzy metric.

Theorem 6.1. [18] Let $(X, M, *)$ be a fuzzy metric space. For $x \in X, r \in (0,1), t > 0$ we define the open ball

$$B(x, r, t) := \{y \in X \ : \ M(x, y, t) > 1 - r\}.$$

Let

$$\mathcal{T}_M := \{T \subset X \ : \ x \in T \text{ iff } (\exists) t > 0, r \in (0,1) \ : \ B(x, r, t) \subseteq T\}.$$

Then $\mathcal{T}_M$ is a topology on X.

Proposition 36. [18] Let (X, d) be a metric space and M_d be the corresponding standard fuzzy metric on X. Then the topology $\mathcal{T}_d$ induced by the metric d, and the topology $\mathcal{T}_{M_d}$ induced by the standard fuzzy metric M_d are the same.

Definition 37. [18] Let $(X, M, *)$ be a fuzzy metric space and (x_n) be a sequence in X. The sequence (x_n) is said to be convergent if there exists $x \in X$ such that $M(x_n, x, t) = 1, (\forall)t > 0$. In this case, x is called the limit of the sequence (x_n) and we write $\lim\limits_{n\to\infty} x_n = x$, or $x_n \to x$.

Remark 38. [18] Let $(X, M, *)$ be a fuzzy metric space. A sequence (x_n) is convergent to x if and only if (x_n) is convergent to x in topology $\mathcal{T}_M$.

Indeed,

$$x_n \to x \text{ in topology } \mathcal{T}_M \Leftrightarrow$$
$$\Leftrightarrow (\forall)r \in (0,1), (\forall)t > 0, (\exists)n_0 \in \mathbb{N} \;:\; x_n \in B(x, r, t), (\forall)n \geq n_0 \Leftrightarrow$$
$$\Leftrightarrow (\forall)r \in (0,1), (\forall)t > 0, (\exists)n_0 \in \mathbb{N} \;:\; M(x_n, x, t) > 1 - r, (\forall)n \geq n_0 \Leftrightarrow$$
$$\Leftrightarrow \lim_{n\to\infty} M(x_n, x, t) = 1, (\forall)t > 0 \;.$$

Definition 39. [18] Let $(X, M, *)$ be a fuzzy metric space and (x_n) be a sequence in X. The sequence (x_n) is said to be a Cauchy sequence if

$$(\forall)r \in (0,1), (\forall)t > 0, (\exists)n_0 \in \mathbb{N} \;:\; M(x_n, x_m, t) > 1 - r, (\forall)n, m \geq n_0 \;.$$

A fuzzy metric in which every Cauchy sequence is convergent is called complete fuzzy metric space.

Definition 40. [18] Let $(X, M, *)$ be a fuzzy metric space. A subset A of X is said to be fuzzy bounded if there exist $r \in (0, 1)$ and $t > 0$ such that $M(x, y, t) > 1 - r$, for all $x, y \in A$.

Remark 41. If $(X, M, *)$ is a fuzzy metric space induced by a metric d on X, then $A \subseteq X$ is fuzzy bounded if and only if A is bounded.

We say that a topological space $(X, \mathcal{T})$ is fuzzy metrizable if the topology is generated by a fuzzy metric. V. Gregori and S. Romaguera [22] proved that a topological space is fuzzy metrizable if and only if it is metrizable.

In paper [21], the fuzzy metric $M^*(x, y, t) := \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ and other fuzzy metrics related to it were studied. This fuzzy metric is useful for measuring perceptual colour differences between colour samples.

7 Fuzzy topological vector spaces

The starting point of the theory of fuzzy topological vector spaces was a series of papers of A.K. Katsaras (see [27], [28], [29]).

Let X be a vector space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$).

Definition 42. [27] Let $\mu_1, \mu_2, \cdots, \mu_n$ be fuzzy sets in X. Then $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ is a fuzzy set in X^n defined by

$$\mu(x_1, x_2, \cdots, x_n) = \mu_1(x_1) \wedge \mu_2(x_2) \wedge \cdots \wedge \mu_n(x_n) \;.$$

Let $f : X^n \to X$, $f(x_1, x_2, \cdots, x_n) = \sum\limits_{k=1}^{n} x_k$. The fuzzy set $f(\mu)$ is called the sum of fuzzy sets $\mu_1, \mu_2, \cdots, \mu_n$ and it is denoted by $\mu_1 + \mu_2 + \cdots + \mu_n$. In fact

$$(\mu_1 + \mu_2 + \cdots + \mu_n)(x) = \vee\{\mu_1(x_1) \wedge \mu_2(x_2) \wedge \cdots \wedge \mu_n(x_n) \;:\; x = \sum_{k=1}^{n} x_k\} \;.$$

Let μ be a fuzzy set in X and $\lambda \in \mathbb{K}$. The fuzzy set $\lambda\mu$ is the image of μ under the map $g : X \to X$, $g(x) = \lambda x$. Thus,

$$(\lambda\mu)(x) = \begin{cases} \mu\left(\frac{x}{\lambda}\right) & \text{if} \quad \lambda \neq 0 \\ 0 & \text{if} \quad \lambda = 0, x \neq 0 \\ \vee\{\mu(y) \ : \ y \in X\} & \text{if} \quad \lambda = 0, x = 0 \end{cases} .$$

Definition 43. [28] A fuzzy topological vector space is a vector space X over $\mathbb{K}$ equipped with a fuzzy topology such that the mappings

$$+ : X \times X \to X \ , \ (x, y) \mapsto x + y$$

$$\cdot : \mathbb{K} \times X \to X \ , \ (\lambda, x) \mapsto \lambda \cdot x$$

are fuzzy continuous when $\mathbb{K}$ has the fuzzy usual topology and $X \times X$ and $\mathbb{K} \times X$ have the corresponding product fuzzy topologies.

In paper [28], the fuzzy vector topologies were characterized in terms of the corresponding families of neighborhoods of zero.

Theorem 7.1. [29] Let X be a vector space over $\mathbb{K}$, and $\mathcal{T}$ be a topology on X. Then $(X, \mathcal{T})$ is a topological vector space if and only if $(X, \omega(\mathcal{T}))$ is a fuzzy topological vector space.

8 Fuzzy normed linear spaces

Studying fuzzy topological vector spaces, A.K. Katsaras [29], introduced in 1984 for the first time, the notion of fuzzy norm on a linear space. In 1992, C. Felbin [17] introduced another concept of fuzzy norm by assigning a fuzzy real number to each element of the linear space. In 1994, S.C. Cheng and J.N. Mordeson [5] introduced another idea of fuzzy norm on a linear space such that their corresponding fuzzy metric was of Kramosil and Michálek type. Following S.C. Cheng and J.N. Mordeson, in 2003, T. Bag and S.K. Samanta [2] introduced a new concept of fuzzy norm, and studied the properties of finite dimensional fuzzy normed linear spaces. A comparative study on fuzzy norms introduced Katsaras, Felbin and Bag and Samanta was made in paper [4]. Other approaches for fuzzy normed linear spaces can be found in [1,7,10,44,48,51,60]. Recently, S. Nădăban introduced the concepts of fuzzy pseudo-norm and fuzzy F-space [11].

Different types of fuzzy bounded linear operators and the relation between fuzzy continuity and fuzzy boundedness were studied in [3], in the context of Bag-Samanta's type fuzzy normed linear spaces. The study of fuzzy continuous mappings and fuzzy bounded linear operators in fuzzy normed linear spaces initiated by T. Bag and S.K. Samanta in [3] was continued by I. Sadeqi and F.S. Kia [51] as well, as S. Nădăban [45] in a more general setting.

Fuzzy bounded linear operators in Felbin's type fuzzy normed linear space were introduced by M. Itoh and M. Cho in [23]. J.Z. Xiao and X.H. Zhu [59,60] gave a new definition for fuzzy norm of bounded operators. In [5], different definitions of strongly fuzzy bounded linear operators and weakly fuzzy bounded linear operators were given and a new idea of their fuzzy norm were introduced. In [25], some properties of the space of all weakly fuzzy bounded linear operators were studied.

In 2006, R. Saadati and J.H. Park introduced the notion of intuitionistic fuzzy Euclidean normed space (see [49], [50]). In paper [42] some special fuzzy norms on $\mathbb{K}^n$ were introduced, and in this way, fuzzy Euclidean normed spaces were obtained, . In order to introduce this concept it is proved that the cartesian product of a finite family of fuzzy normed linear spaces is a fuzzy normed linear space.

Definition 44. [27] A fuzzy set ρ in X is said to be:

1. convex if $t\rho + (1-t)\rho \subseteq \rho, (\forall)t \in [0,1]$;
2. balanced if $\lambda\rho \subseteq \rho, (\forall)\lambda \in \mathbb{K}, |\lambda| \leq 1$;
3. absorbing if $\bigvee_{t>0} t\rho = 1$;
4. absolutely convex if it is both convex and balanced.

Proposition 45. [27] Let ρ be a fuzzy set in X. Then:

1. ρ is convex if and only if

$$\rho(tx + (1-t)y) \geq \rho(x) \wedge \rho(y), (\forall)x, y \in X, (\forall)t \in [0,1];$$

2. ρ is balanced if and only if $\rho(\lambda x) \geq \rho(x), (\forall)x \in X, (\forall)\lambda \in \mathbb{K}, |\lambda| \leq 1$.

Definition 46. [29] A Katsaras fuzzy semi-norm on X is a fuzzy set ρ in X which is absolutely convex and absorbing.

Proposition 47. [31] Let ρ be a Katsaras fuzzy semi-norm on X. Let

$$p_\alpha(x) := \inf\{t > 0 \ : \ \rho\left(\frac{x}{t}\right) > \alpha\}, \alpha \in (0,1) \ .$$

Then $\mathcal{P} = \{p_\alpha\}_{\alpha \in (0,1)}$ is an ascending family of semi-norms on X.

Definition 48. [10] A fuzzy semi-norm ρ on X will be called Katsaras fuzzy norm if

$$\rho\left(\frac{x}{t}\right) = 1, (\forall)t > 0 \ \Rightarrow \ x = 0 \ .$$

Remark 49. a) It is easy to see that

$$\left[\rho\left(\frac{x}{t}\right) = 1, (\forall)t > 0 \Rightarrow x = 0\right] \Leftrightarrow \left[\inf_{t>0} \rho\left(\frac{x}{t}\right) < 1, \text{for } x \neq 0\right] \ .$$

b) The condition $\left[\rho\left(\frac{x}{t}\right) = 1, (\forall)t > 0 \Rightarrow x = 0\right]$ is much weaker than that one imposed by A.K. Katsaras [29],

$$\left[\inf_{t>0} \rho\left(\frac{x}{t}\right) = 0, \text{for } x \neq 0\right] \ .$$

Proposition 50. [10] Let ρ be a Katsaras fuzzy semi-norm and

$$p_\alpha(x) := \inf\{t > 0 \ : \ \rho\left(\frac{x}{t}\right) > \alpha\}, \alpha \in (0,1).$$

Then the family of semi-norms $\mathcal{P} = \{p_\alpha\}_{\alpha \in (0,1)}$ is sufficient if and only if ρ is a Katsaras fuzzy norm.

Definition 51. [17] Let X be a vector space over $\mathbb{R}$, let $||\cdot|| : X \to \mathbb{R}^*(I)$ and let the mappings $L, R : [0,1]\times[0,1] \to [0,1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$. We write $[||x||]_\alpha = [||x||_1^\alpha, ||x||_2^\alpha]$, for $x \in X, \alpha \in (0,1]$.

We suppose that $(\forall)x \in X, x \neq 0$ there exists $\alpha_0 \in (0,1]$ independent of x such that for all $\alpha \leq \alpha_0$ we have

$$(A) \ ||x||_2^\alpha < \infty \ ,$$

$$(B) \ \inf ||x||_1^\alpha > 0 \ .$$

The quadruple $(X, ||\cdot||, L, R)$ is called fuzzy normed linear space and $||\cdot||$ a fuzzy norm, if

1. $||x|| = 0$ if and only if $x = 0$;
2. $||rx|| = |r| \cdot ||x||, (\forall)x \in X, r \in \mathbb{R}$;
3. for all $x, y \in X$,
   (a) whenever $s \leq ||x||_1^1, t \leq ||y||_1^1$ and $s + t \leq ||x + y||_1^1$,
   $$||x + y||(s + t) \geq L(||x||(s), ||y||(t)) ,$$
   (b) whenever $s \geq ||x||_1^1, t \geq ||y||_1^1$ and $s + t \geq ||x + y||_1^1$,
   $$||x + y||(s + t) \leq R(||x||(s), ||y||(t)) .$$

Remark 52. C. Felbin [17] proved that, if $L(x, y) = \min\{x, y\}$ and $R(x, y) = \max\{x, y\}$, then the triangle inequality (3) in previous definition is equivalent to $||x + y|| \leq ||x|| + ||y||$. Further $|| \cdot ||_\alpha^i$ are crisp norms on X, for each $\alpha \in (0, 1]$ and $i = 1, 2$.

Remark 53. In paper [5], Felbin's definition of fuzzy normed linear space is slightly modified in the sense that:

1. the value of the fuzzy norm is taken to be a fuzzy real number in the sense of J.Z. Xiao and X.H. Zhu [59];
2. the condition (A) and (B) of Felbin's definition are relaxed by the condition
$$(A\prime)\ x \neq 0 \Rightarrow ||x||(t) = 0, (\forall)t \leq 0 .$$

Definition 54. [10] Let X be a vector space over a field $\mathbb{K}$ and $*$ be a continuous t-norm. A fuzzy set N in $X \times [0, \infty)$ is called a fuzzy norm on X if it satisfies:

(N1) $N(x, 0) = 0, (\forall)x \in X$;

(N2) $[N(x, t) = 1, (\forall)t > 0]$ if and only if $x = 0$;

(N3) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right), (\forall)x \in X, (\forall)t \geq 0, (\forall)\lambda \in \mathbb{K}^*$;

(N4) $N(x + y, t + s) \geq N(x, t) * N(y, s), (\forall)x, y \in X, (\forall)t, s \geq 0$;

(N5) $(\forall)x \in X$, $N(x, \cdot)$ is left continuous and $\lim\limits_{t\to\infty} N(x, t) = 1$.

The triple $(X, N, *)$ will be called fuzzy normed linear space (briefly FNL-space).

Remark 55. a) T. Bag and S.K. Samanta [2], [3] gave a similar definition for $* = \wedge$, but in order to obtain some important results they assumed that the fuzzy norm satisfies also the following conditions:

(N6) $N(x, t) > 0, (\forall)t > 0 \Rightarrow x = 0$;

(N7) $(\forall)x \neq 0, N(x, \cdot)$ is a continuous function and strictly increasing on the subset $\{t : 0 < N(x, t) < 1\}$ of $\mathbb{R}$.

The results obtained by T. Bag and S.K. Samanta can be found in these more general settings [10].

b) I. Goleţ [7], C. Alegre and S. Romaguera [1] gave also the same definition in the context of real vector spaces.

Remark 56. $N(x, \cdot)$ is nondecreasing, $(\forall)x \in X$.

Theorem 8.1. [10] If $(X, N, *)$ is a FNL-space, then

$$M : X \times X \times [0, \infty) \to [0, 1], M(x, y, t) = N(x - y, t)$$

is a fuzzy metric on X, which is called the fuzzy metric induced by the fuzzy norm N. Moreover, we have:

1. M is a translation-invariant fuzzy metric;
2. $M(\lambda x, \lambda y, t) = M\left(x, y, \frac{t}{|\lambda|}\right), (\forall) x \in X, (\forall) t \geq 0, (\forall) \lambda \in \mathbb{K}^*$.

Corollary 57. [10] Let $(X, N, *)$ be a FNL-space. For $x \in X, r \in (0, 1), t > 0$ we define the open ball

$$B(x, r, t) := \{y \in X \ : \ N(x - y, t) > 1 - r\} .$$

Then

$$\mathcal{T}_N := \{T \subset X \ : \ x \in T \text{ iff } (\exists) t > 0, r \in (0, 1) \ : \ B(x, r, t) \subseteq T\}$$

is a topology on X.

Moreover, if the t-norm $*$ satisfies $\sup\limits_{x \in (0,1)} x * x = 1$, then $(X, \mathcal{T}_N)$ is Hausdorff.

Theorem 8.2. [10] Let $(X, N, *)$ be a FNL-space. Then $(X, \mathcal{T}_N)$ is a metrizable topological vector space.

9 Conclusions

Lotfi A. Zadeh, born on February 4, 1921, is a famous mathematician, electrical engineer, computer scientist, and Professor Emeritus at the University of California, Berkeley, United State of America. He is father of fuzzy sets, fuzzy logic and computing with words. His pioneering paper, entitled "Fuzzy Sets" (1965, [61]), is cited over 58,540 time in many prestigious journals, and all his papers are cited over 151,300 time.

Some scientists, especially philosophers and mathematicians, had attempted to formalize the process of logical deduction. Their work culminated in the invention of the programmable digital computer, a machine based on the abstract essence of mathematical reasoning. This machine and the ideas behind it inspired a handful of scientists to begin seriously discussing the possibility of building an artificial brain.

In this survey paper we mentioned some fuzzy mathematical structures as fuzzy real line, fuzzy topological spaces, fuzzy metric spaces, fuzzy topological vector spaces, fuzzy normed linear spaces and fuzzy F-space.

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(a) Lotfi A. Zadeh & Ioan Dzitac at ICCCC 2008 (b) Ed. by L.A. Zadeh, D. Tufis, F.G. Filip, I. Dzitac

Figure 1: Meeting with Professor Lotfi A. Zadeh (Agora University of Oradea, Romania, 2008)

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Group Decision Making with Incomplete Interval-valued Fuzzy Preference Relations Based on the Minimum Operator

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Abstract: This paper presents a new method to estimate the unknown values in incomplete interval-valued fuzzy preference relations (IVFPRs). The method is based on the min-consistency and is used to develop the algorithm for group decision making (GDM) dealing with incomplete IVFPRs.
Keywords: fuzzy preference relation, interval-valued fuzzy preference relation; incomplete interval-valued fuzzy preference relation; min-consistency; group decision making.

1 Introduction and preliminaries

GDM is a situation faced when a number of experts work together to find the best alternative(s) from a set of feasible alternatives. Each expert may have exclusive inspirations or objectives and a different decision procedure, but has a common interest in approaching to select the "best" option(s). Preference relation is the most common representation format used in GDM because it is a valuable tool in modeling decision processes, when we have to combine experts' preferences into group preferences [6, 14, 15]. In a preference relation an expert assigns a numerical value to every pair of alternatives that reflects some degree of preference of the first alternative over the second alternative. Mainly two types of preference relations have been used to develop the decision models; multiplicative preference relations (MPRs) [2, 14], and fuzzy preference relations (FPRs) [6, 16].

The popular preference relations, which are being used to express an expert's preferences over alternatives, are FPRs. In a decision making procedure, an expert mostly needs to compare a finite set of alternatives x_i $(i = 1, 2, ..., n)$ and construct an FPR [6, 13, 16, 17]. However, an expert may have imprecise information for the preference degrees of one alternative over another and it may not always be possible to estimate his/her preference by means of an exact numerical value. In such a situation, an expert constructs an IVFPR.

In 2004, Z. S. Xu defined the notion of compatibility degree of two IVFPRs and showed the compatible connection among individual and collective IVFPRs [18]. In 2005, F. Herrera et al. established an aggregation process for combining IVFPRs with other forms of information as; numerical preference relation (NPR) and linguistic preference relation (LPR) [7]. In 2007, Y. Jiang proposed a technique to measure the similarity degree of two IVFPRs and used the error-propagation rule to find the priority vector of the accumulated IVFPRs [8]. In 2008, Z. S. Xu and Chen developed some linear programming models to derive the priority weights from several IVFPRs [20].

All the above researches focused on the IVFPRs with complete information. However, in DM problems such situations are unavoidable in which an expert does not have comprehensive information of the problem because of time constraint, lack of knowledge and the expert's limited expertise within the problem domain [1, 3, 5, 10, 19, 22, 24, 33]. Consequently, the expert may not be able to give his/her opinion about specific traits of the problem, and hence an incomplete preference relation would be constructed. In literature, researches based on incomplete FPRs have been given, but there are only few researches in GDM related to incomplete IVFPRs [23].

In this paper, a new technique for GDM by using incomplete IVFPRs is developed. Obviously, the consistent information is more applicable or important than the information having ambiguities, consistency is linked with definite transitivity properties. Several properties have been endorsed to model transitivity of FPRs, one of these properties is the max-min transitivity. In this paper, a procedure, based on min-transitivity property is proposed to determine unknown interval-valued preferences of one alternative over others and further, it is extended to develop an algorithm for GDM to select the best alternative.

Definition 1.1. [4] An interval-valued fuzzy set A on a universe X is defined as:

$$A = \{(a, [x^-, x^+])|a \in X,\ [x^-, x^+] \in L([0, 1])\}$$

where $L([0, 1]) = \{[x^-, x^+]|[x^-, x^+] \subseteq [0, 1] \text{ with } x^- \leq x^+\}$.

Arithmetic operations can be performed on closed intervals . The following formulae can be used for all $P, Q \in L([0, 1])$ ($P = [p^-, p^+]$ and $Q = [q^-, q^+]$) [12]:

- $P + Q = [p^- + q^-, p^+ + q^+]$,
- $P - Q = [p^- - q^+, p^+ - q^-]$,
- $P \cdot Q = [\min(p^-q^-, p^-q^+, p^+q^-, p^+q^+), \max(p^-q^-, p^-q^+, p^+q^-, p^+q^+)]$,
- $P/Q = [p^-, p^+] \cdot [\frac{1}{q^+}, \frac{1}{q^-}]$ if $0 \notin [q^-, q^+]$.

Definition 1.2. [4] Let X be a universe and A and B two interval-valued fuzzy sets. The inclusion of A into B is defined as: $A \subseteq B$ if and only if $A(a) \subseteq B(a)$ for all $a \in X$ and the equality between A and B is defined as: $A = B$ if and only if $A(a) = B(a)$ for all $a \in X$.

Definition 1.3. [9] A triangular norm (t-norm) T is an increasing, associative, commutative and $[0, 1] \times [0, 1] \to [0, 1]$ mapping satisfying: $T(1, x) = x$ for all $x \in [0, 1]$.

The t-norm to be used in this paper is $T(x, y) = \min(x, y)$. The concept of a t-norm on $[0, 1]$ can be extended to subintervals of $[0, 1]$.

Definition 1.4. An extended t-norm, T_e, is an increasing, commutative, associative and $L([0, 1]) \times L([0, 1]) \to L([0, 1])$ mapping that satisfies:

$$T_e([1, 1], [x^-, x^+]) = [x^-, x^+] \text{ for all } [x^-, x^+] \in L([0, 1]).$$

Let T be a triangular norm. The mapping T_e defined as:

$$T_e([a^-, a^+], [b^-, b^+]) = [T(a^-, b^-), T(a^+, b^+)]$$

for $[a^-, a^+]$, $[b^-, b^+] \in L([0, 1])$, is an extended t-norm on $(L([0, 1]), \subseteq)$, where $\subseteq$ represents the crisp set inclusion.

The extended interval t-norm corresponding to the minimum-operator can be computed by:

$$T_{\min}([a^-, a^+], [b^-, b^+]) = [\min(a^-, b^-), \min(a^+, b^+)]. \tag{1}$$

Definition 1.5. [15] A fuzzy preference relation R over a finite set X of alternatives, $X = \{x_1, x_2, x_3, ..., x_n\}$, is a fuzzy set on the product set $X \times X$, i.e., it is characterized by a membership function $\mu_R : X \times X \to [0, 1]$.

According to Definition 1.5, a fuzzy preference relation R on X can be conveniently expressed by an $n \times n$ matrix $R = (r_{ij})_{n\times n}$, where r_{ij} denotes the degree of preference of alternative x_i over the alternative x_j with $r_{ij} \in [0, 1]$, $r_{ii} = 0.5$, $r_{ij} + r_{ji} = 1$ (additive reciprocity) for $1 \le i \le n$ and $1 \le j \le n$. If $r_{ij} = 0.5$, then there is no difference between the alternatives x_i and x_j. If $r_{ij} > 0.5$, then alternative x_i is preferred over the alternative x_j. If $r_{ij} = 1$, then the alternative x_i is definitely preferred over the alternative x_j.

Definition 1.6. [18] Let $\overline{R} = (\overline{r}_{ij})_{n\times n}$ be a fuzzy preference relation over the set of alternatives $X = \{x_1, x_2, x_3, ..., x_n\}$ where $\overline{r}_{ij} = [r^-_{ij}, r^+_{ij}]$, $0 \le r^-_{ij} \le r^+_{ij} \le 1$, $\overline{r}_{ij} = [1, 1] - \overline{r}_{ji}$ and $\overline{r}_{ii} = [0.5, 0.5]$ for all $i, j \in N$, then $\overline{R}$ is called an interval-valued fuzzy preference relation.

Definition 1.7. An IVFPR $\overline{R}$ is said to be min-consistent, if for all i, j and k belonging to $\{1, 2, 3, ..., n\}$ it holds:

$$\overline{r}_{ik} \ge T_{\min}(\overline{r}_{ij}, \overline{r}_{jk}) \qquad \text{(min-transitivity)}.$$

Definition 1.8. An IVFPR relation $\overline{R} = (\overline{r}_{ij})_{n\times n}$ is said to be incomplete if it contains at least one unknown preference value $\overline{r}_{ij}$ for which the expert has no idea about the degree of preference of alternative x_i over the alternative x_j.

2 Method to repair an incomplete IVFPR

This section presents a new technique to estimate missing values in an incomplete IVFPR. Further, the algorithm is used to construct a min-consistent matrix. In order to determine unknown values in an incomplete IVFPR $\overline{R} = (\overline{r}_{ij})_{n\times n}$, the pairs of alternatives for known and unknown preference values are represented by the following sets:

$$K_P = \{(i, j)|\overline{r}_{ij} \text{ is known}\}, \tag{2}$$
$$U_P = \{(i, j)|\overline{r}_{ij} \text{ is unknown}\}, \tag{3}$$

where the preference value of alternative x_i over x_j belongs to the family of closed subintervals of $[0, 1]$ (i.e., $\overline{r}_{ij} \in L([0, 1])$). Since $\overline{r}_{ij} = [1, 1] - \overline{r}_{ji}$, $\overline{r}_{ii} = [0.5, 0.5]$ for $1 \le i \le n$ and $1 \le j \le n$, therefore, the min-transitivity of definition 1.7 can be written as:

$$\overline{r}_{ik} \ge T_{\min}(\overline{r}_{ij}, \overline{r}_{jk});\ \overline{r}_{ik} \ge T_{\min}(1 - \overline{r}_{ji}, \overline{r}_{jk});\ \overline{r}_{ik} \ge T_{\min}(\overline{r}_{ij}, 1 - \overline{r}_{kj}). \tag{4}$$

Hence, the following sets can be defined to determine the unknown preference value $\overline{r}_{ik}$ of alternative x_i over alternative x_k:

$$S^1_{ik} = \{j|(i, j) \in K_P, (j, k) \in K_P \text{ and } (i, k) \in U_P\}, \tag{5}$$
$$S^2_{ik} = \{j|(j, i) \in K_P, (j, k) \in K_P \text{ and } (i, k) \in U_P\}, \tag{6}$$
$$S^3_{ik} = \{j|(i, j) \in K_P, (k, j) \in K_P \text{ and } (i, k) \in U_P\}, \tag{7}$$

for $i = \{1, 2, 3, ..., n\}$, $j = \{1, 2, 3, ..., n\}$ and $k = \{1, 2, 3, ..., n\}$. Based on (5),(6) and (7), we can determine the unknown preference value $\overline{r}_{ik}$ for x_i over x_k as follows:

$$\overline{r}_{ik} = \frac{\overline{r}^1_{ik} + \overline{r}^2_{ik} + \overline{r}^3_{ik}}{3}, \tag{8}$$

where

$$\overline{r}_{ik}^{1} = \begin{cases} \frac{1}{|S_{ik}^{1}|} \sum_{j \in S_{ik}^{1}} T_{\min}(\overline{r}_{ij}, \overline{r}_{jk}), & \text{if } |S_{ik}^{1}| \neq 0 \\ [0.5, 0.5], & \text{otherwise} \end{cases} \tag{9}$$

$$\overline{r}_{ik}^{2} = \begin{cases} \frac{1}{|S_{ik}^{2}|} \sum_{j \in S_{ik}^{2}} T_{\min}([1,1] - \overline{r}_{ji}, \overline{r}_{jk}), & \text{if } |S_{ik}^{2}| \neq 0 \\ [0.5, 0.5], & \text{otherwise} \end{cases} \tag{10}$$

$$\overline{r}_{ik}^{3} = \begin{cases} \frac{1}{|S_{ik}^{3}|} \sum_{j \in S_{ik}^{3}} T_{\min}(\overline{r}_{ij}, [1,1] - \overline{r}_{kj}), & \text{if } |S_{ik}^{3}| \neq 0 \\ [0.5, 0.5], & \text{otherwise} \end{cases} \tag{11}$$

where $|S_{ik}^{1}|, |S_{ik}^{2}|$ and $|S_{ik}^{3}|$ are the cardinalities of the sets S_{ik}^{1}, S_{ik}^{2} and S_{ik}^{3} respectively.

$$K_P^{'} = K_P \cup \{(i,k)\}, \tag{12}$$

$$U_P^{'} = U_P - \{(i,k)\}. \tag{13}$$

To achieve min-consistency of the IVFPR $\overline{R}$, following scaling conditions will be used:

(i) If $r_{ij}^{-} + r_{ji}^{+} < 1$ and $r_{ij}^{+} + r_{ji}^{-} < 1$, then

$$\overline{r}_{ij} = \left[r_{ij}^{-} + \frac{1 - (r_{ij}^{-} + r_{ji}^{+})}{2}, r_{ij}^{+} + \frac{1 - (r_{ij}^{+} + r_{ji}^{-})}{2} \right] \tag{14}$$

and

$$\overline{r}_{ji} = \left[r_{ji}^{-} + \frac{1 - (r_{ij}^{+} + r_{ji}^{-})}{2}, r_{ji}^{+} + \frac{1 - (r_{ij}^{-} + r_{ji}^{+})}{2} \right]. \tag{15}$$

(ii) If $r_{ij}^{-} + r_{ji}^{+} < 1$ and $r_{ij}^{+} + r_{ji}^{-} > 1$, then

$$\overline{r}_{ij} = \left[r_{ij}^{-} + \frac{1 - (r_{ij}^{-} + r_{ji}^{+})}{2}, r_{ij}^{+} - \frac{r_{ij}^{+} + r_{ji}^{-} - 1}{2} \right] \tag{16}$$

and

$$\overline{r}_{ji} = \left[r_{ji}^{-} - \frac{r_{ij}^{+} + r_{ji}^{-} - 1}{2}, r_{ji}^{+} + \frac{1 - (r_{ij}^{-} + r_{ji}^{+})}{2} \right]. \tag{17}$$

(iii) If $r_{ij}^{-} + r_{ji}^{+} > 1$ and $r_{ij}^{+} + r_{ji}^{-} < 1$, then

$$\overline{r}_{ij} = \left[r_{ij}^{-} - \frac{r_{ij}^{-} + r_{ji}^{+} - 1}{2}, r_{ij}^{+} + \frac{1 - (r_{ij}^{+} + r_{ji}^{-})}{2} \right] \tag{18}$$

and

$$\overline{r}_{ji} = \left[r_{ji}^{-} + \frac{1 - (r_{ij}^{+} + r_{ji}^{-})}{2}, r_{ji}^{+} - \frac{r_{ij}^{-} + r_{ji}^{+} - 1}{2} \right]. \tag{19}$$

(iv) If $r_{ij}^{-} + r_{ji}^{+} > 1$ and $r_{ij}^{+} + r_{ji}^{-} > 1$, then

$$\overline{r}_{ij} = \left[r_{ij}^{-} - \frac{r_{ij}^{-} + r_{ji}^{+} - 1}{2}, r_{ij}^{+} - \frac{r_{ij}^{+} + r_{ji}^{-} - 1}{2} \right] \tag{20}$$

and

$$\overline{r}_{ji} = \left[r_{ji}^- - \frac{r_{ij}^+ + r_{ji}^- - 1}{2}, r_{ji}^+ - \frac{r_{ij}^- + r_{ji}^+ - 1}{2} \right]. \tag{21}$$

Example 2.1. Let $\overline{R} = (\overline{r}_{ij})_{4\times 4}$ be an incomplete IVFPR for the alternatives x_1, x_2, x_3 and x_4, given as follows:

$$\overline{R} = \begin{bmatrix} [0.5, 0.5] & \overline{r}_{12} & [0.4, 0.6] & [0.3, 0.7] \\ \overline{r}_{21} & [0.5, 0.5] & [0.7, 0.8] & \overline{r}_{24} \\ [0.4, 0.6] & [0.2, 0.3] & [0.5, 0.5] & [0.3, 0.4] \\ [0.3, 0.7] & \overline{r}_{42} & [0.6, 0.7] & [0.5, 0.5] \end{bmatrix}$$

where $\overline{r}_{12}, \overline{r}_{21}, \overline{r}_{24}$ and $\overline{r}_{42}$ are unknown preference values. Now applying (2)-(13) to estimate the unknown preference values for the alternative x_i over x_k, $1 \le i \le 4$ and $1 \le k \le 4$, we obtain:

$$\begin{aligned}
K_P &= \{(1,1),(1,3),(1,4),(2,2),(2,3),(3,1),(3,2),(3,3),(3,4),(4,1),\\
&\quad\ (4,3),(4,4)\},\\
U_P &= \{(1,2),(2,1),(2,4),(4,2)\}.\\
S_{12}^1 &= \{3\}, \qquad S_{12}^2 = \{3\}, \qquad S_{12}^3 = \{3\},\\
\overline{r}_{12}^1 &= T_{\min}(\overline{r}_{13}, \overline{r}_{32}) = T_{\min}([0.4,0.6],[0.2,0.3]) = [0.2,0.3],\\
\overline{r}_{12}^2 &= T_{\min}([1,1]-\overline{r}_{31}, \overline{r}_{32}) = T_{\min}([0.4,0.6],[0.2,0.3]) = [0.2,0.3],\\
\overline{r}_{12}^3 &= T_{\min}(\overline{r}_{13}, [1,1]-\overline{r}_{23}) = T_{\min}([0.4,0.6],[0.2,0.3]) = [0.2,0.3],\\
\overline{r}_{12} &= \frac{1}{3}(\overline{r}_{12}^1 + \overline{r}_{12}^2 + \overline{r}_{12}^3) = [0.2,0.3].\\
K_P' &= \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(3,1),(3,2),(3,3),(3,4),\\
&\quad\ (4,1),(4,3),(4,4)\},\\
U_P' &= U_P - \{(1,2)\} = \{(2,1),(2,4),(4,2)\}.\\
S_{21}^1 &= \{3\}, \qquad S_{21}^2 = \{1,3\}, \qquad S_{21}^3 = \{2,3\},\\
\overline{r}_{21}^1 &= T_{\min}(\overline{r}_{23}, \overline{r}_{31}) = T_{\min}([0.7,0.8],[0.4,0.6]) = [0.4,0.6],\\
\overline{r}_{21}^2 &= \frac{1}{2}[T_{\min}([1,1]-\overline{r}_{12}, \overline{r}_{11}) + T_{\min}([1,1]-\overline{r}_{32}, \overline{r}_{31})]\\
&= \frac{1}{2}[T_{\min}([0.7,0.8],[0.5,0.5]) + T_{\min}([0.7,0.8],[0.4,0.6])]\\
&= \frac{1}{2}[[0.5,0.5]+[0.4,0.6]] = [0.45,0.55],
\end{aligned}$$

$$
\begin{aligned}
\overline{r}_{21}^{3} &= \frac{1}{2}[T_{\min}(\overline{r}_{22},[1,1]-\overline{r}_{12})+T_{\min}(\overline{r}_{23},[1,1]-\overline{r}_{13})] \\
&= \frac{1}{2}[T_{\min}([0.5,0.5],[0.7,0.8])+T_{\min}([0.7,0.8],[0.4,0.6]] \\
&= \frac{1}{2}[[0.5,0.5]+[0.4,0.6]]=[0.45,0.55], \\
\overline{r}_{21} &= \frac{1}{3}(\overline{r}_{21}^{1}+\overline{r}_{21}^{2}+\overline{r}_{21}^{3})=[0.433,0.567]. \\
K_P^{''} &= \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(3,4), \\
&\quad (4,1),(4,3),(4,4)\}, \\
U_P^{''} &= U^{'}-\{(2,1)\}=\{(2,4),(4,2)\}. \\
S_{24}^{1} &= \{1,3\}, \qquad S_{24}^{2}=\{1,3\}, \qquad S_{24}^{3}=\{1,3\}, \\
\overline{r}_{24}^{1} &= \frac{1}{2}[T_{\min}(\overline{r}_{21},\overline{r}_{14})+T_{\min}(\overline{r}_{23},\overline{r}_{34})] \\
&= \frac{1}{2}[T_{\min}([0.433,0.567],[0.3,0.7])+T_{\min}([0.7,0.8],[0.3,0.4])] \\
&= \frac{1}{2}[[0.3,0.567]+[0.3,0.4]]=[0.3,0.484], \\
\overline{r}_{24}^{2} &= \frac{1}{2}[T_{\min}([1,1]-\overline{r}_{12},\overline{r}_{14})+T_{\min}([1,1]-\overline{r}_{32},\overline{r}_{34})] \\
&= \frac{1}{2}[T_{\min}([0.7,0.8],[0.3,0.7])+T_{\min}([0.7,0.8],[0.3,0.4])] \\
&= \frac{1}{2}[[0.3,0.7]+[0.3,0.4]]=[0.3,0.55], \\
\overline{r}_{24}^{3} &= \frac{1}{2}[T_{\min}(\overline{r}_{21},[1,1]-\overline{r}_{41})+T_{\min}(\overline{r}_{23},[1,1]-\overline{r}_{43})] \\
&= \frac{1}{2}[T_{\min}([0.433,0.567],[0.3,0.7])+T_{\min}([0.7,0.8],[0.3,0.4])] \\
&= \frac{1}{2}[[0.3,0.567]+[0.3,0.4]]=[0.3,0.484], \\
\overline{r}_{24} &= \frac{1}{3}(\overline{r}_{24}^{1}+\overline{r}_{24}^{2}+\overline{r}_{24}^{3})=[0.3,0.506]. \\
K_P^{'''} &= \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3), \\
&\quad (3,4),(4,1),(4,3),(4,4)\}, \\
U_P^{'''} &= U^{''}-\{(2,4)\}=\{(4,2)\}. \\
S_{42}^{1} &= \{1,3\}, \qquad S_{42}^{2}=\{1,2,3\}, \qquad S_{42}^{3}=\{1,3,4\}, \\
\overline{r}_{42}^{1} &= \frac{1}{2}[T_{\min}(\overline{r}_{41},\overline{r}_{12})+T_{\min}(\overline{r}_{43},\overline{r}_{32})] \\
&= \frac{1}{2}[T_{\min}([0.3,0.7],[0.2,0.3])+T_{\min}([0.6,0.7],[0.2,0.3])] \\
&= \frac{1}{2}[[0.2,0.3]+[0.2,0.3]]=[0.2,0.3], \\
\overline{r}_{42}^{2} &= \frac{1}{3}[T_{\min}([1,1]-\overline{r}_{14},\overline{r}_{12})+T_{\min}([1,1]-\overline{r}_{24},\overline{r}_{22})+T_{\min}([1,1]-\overline{r}_{34},\overline{r}_{32})] \\
&= \frac{1}{3}[[0.2,0.3]+[0.5,0.494]+[0.2,0.3]] \\
&= [0.298,0.367],
\end{aligned}
$$

$$\begin{aligned}
\overline{r}_{42}^{3} &= \frac{1}{3}[T_{\min}(\overline{r}_{41},[1,1]-\overline{r}_{21})+T_{\min}(\overline{r}_{43},[1,1]-\overline{r}_{23})+T_{\min}(\overline{r}_{44},[1,1]-\overline{r}_{24})] \\
&= \frac{1}{3}[[0.3,0.567]+[0.2,0.3]+[0.494,0.5]] \\
&= [0.331,0.456], \\
\overline{r}_{42} &= \frac{1}{3}(\overline{r}_{42}^{1}+\overline{r}_{42}^{2}+\overline{r}_{42}^{3}) = [0.276,0.374]. \\
K_P^{''''} &= \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3), \\
& \quad (3,4),(4,1),(4,2),(4,3),(4,4)\}, \\
U_P^{''''} &= \phi.
\end{aligned}$$

Hence, the complete IVFPR is

$$\overline{R} = \begin{bmatrix} [0.5,0.5] & [0.2,0.3] & [0.4,0.6] & [0.3,0.7] \\ [0.433,0.567] & [0.5,0.5] & [0.7,0.8] & [0.3,0.506] \\ [0.4,0.6] & [0.2,0.3] & [0.5,0.5] & [0.3,0.4] \\ [0.3,0.7] & [0.276,0.374] & [0.6,0.7] & [0.5,0.5] \end{bmatrix} \tag{22}$$

By applying scaling condition on (18), $\overline{R}$ becomes a min-consistent IVFPR $\widetilde{\overline{R}}$ as follows:

$$\widetilde{\overline{R}} = \begin{bmatrix} [0.5,0.5] & [0.316,0.434] & [0.4,0.6] & [0.3,0.7] \\ [0.566,0.684] & [0.5,0.5] & [0.7,0.8] & [0.463,0.615] \\ [0.4,0.6] & [0.2,0.3] & [0.5,0.5] & [0.3,0.4] \\ [0.3,0.7] & [0.385,0.537] & [0.6,0.7] & [0.5,0.5] \end{bmatrix}.$$

3 A new algorithm to choose the best alternative in GDM with incomplete IVFPRs.

In this section, a new algorithm is presented for GDM with incomplete IVFPRs by using min-consistency. An explanatory example is given to validate the anticipated technique. For ease, the structure of the determination process is also shown in Figure1. Suppose that there are n alternatives $x_1, x_2, ..., x_n$ and m experts $E_1, E_2, ..., E_m$. Let $\overline{R}^q$ be the IVFPR for the expert E_q shown as follows:

$$\overline{R}^q = \left(\overline{r}_{ij}^q\right)_{n\times n} = \begin{bmatrix} [0.5,0.5] & \overline{r}_{12}^q & . & . & \overline{r}_{1n}^q \\ \overline{r}_{21}^q & [0.5,0.5] & . & . & \overline{r}_{2n}^q \\ . & . & . & & . \\ . & . & & . & . \\ \overline{r}_{n1}^q & \overline{r}_{n2}^q & . & . & [0.5,0.5] \end{bmatrix}, \tag{23}$$

where $\overline{r}_{ij}^q \in L([0,1])$ is the preference value given by expert E_q for alternative x_i over x_j, $\overline{r}_{ij}^q = [1,1]-\overline{r}_{ji}^q$, $\overline{r}_{ii}^q = [0.5,0.5]$, $1 \le i \le n, 1 \le j \le n$ and $1 \le q \le m$. The proposed GDM technique is given as follows:

Step 1: Determine the sets K_P^q and U_P^q of pairs of alternatives for known and unknown preference values respectively, shown as follows:

$$K_P^q = \{(i,j)|\overline{r}_{ij}^q \text{ is known}\}, \tag{24}$$
$$U_P^q = \{(i,j)|\overline{r}_{ij}^q \text{ is unknown}\}, \tag{25}$$

where $1 \leq i \leq n, 1 \leq j \leq n$ and $1 \leq q \leq m$.
Step 2: If $U = \phi$, then skip **Step 2**, otherwise construct the sets S_{ik}^{q1}, S_{ik}^{q2} and S_{ik}^{q3} based on the sets K_P^q and U_P^q. The constructed sets are used to estimate the unknown preference values $\overline{r}_{ik}^q$ for the alternative x_i over x_k by expert E_q as follows:

$$\overline{r}_{ik} = \frac{\overline{r}_{ik}^{q1} + \overline{r}_{ik}^{q2} + \overline{r}_{ik}^{q3}}{3}, \tag{26}$$

$$\begin{aligned}
S_{ik}^{q1} &= \{j|(i,j) \in K_P^q, (j,k) \in K_P^q \text{ and } (i,k) \in U_P^q\}, && (27)\\
S_{ik}^{q2} &= \{j|(j,i) \in K_P^q, (j,k) \in K_P^q \text{ and } (i,k) \in U_P^q\}, && (28)\\
S_{ik}^{q3} &= \{j|(i,j) \in K_P^q, (k,j) \in K_P^q \text{ and } (i,k) \in U_P^q\}, && (29)
\end{aligned}$$

$$\overline{r}_{ik}^{q1} = \begin{cases} \frac{1}{|S_{ik}^{q1}|} \sum\limits_{j \in S_{ik}^{q1}} T_{\min}(\overline{r}_{ij}^q, \overline{r}_{jk}^q), & \text{if } |S_{ik}^{q1}| \neq 0 \\ [0.5, 0.5], & \text{otherwise} \end{cases} \tag{30}$$

$$\overline{r}_{ik}^{q2} = \begin{cases} \frac{1}{|S_{ik}^{q2}|} \sum\limits_{j \in S_{ik}^{q2}} T_{\min}([1,1] - \overline{r}_{ij}^q, \overline{r}_{jk}^q), & \text{if } |S_{ik}^{q2}| \neq 0 \\ [0.5, 0.5], & \text{otherwise} \end{cases} \tag{31}$$

$$\overline{r}_{ik}^{q3} = \begin{cases} \frac{1}{|S_{ik}^{q3}|} \sum\limits_{j \in S_{ik}^{q3}} T_{\min}(\overline{r}_{ij}^q, [1,1] - \overline{r}_{jk}^q), & \text{if } |S_{ik}^{q3}| \neq 0 \\ [0.5, 0.5], & \text{otherwise} \end{cases} \tag{32}$$

where $|S_{ik}^{q1}|, |S_{ik}^{q2}|$ and $|S_{ik}^{q3}|$ are the cardinalities of the sets S_{ik}^{q1}, S_{ik}^{q2} and S_{ik}^{q3} respectively.

$$\begin{aligned}
K_P^{'q} &= K_P^q \cup \{(i,k)\}, && (33)\\
U_P^{'q} &= U_P^q - \{(i,k)\}. && (34)
\end{aligned}$$

Step 3: To satisfy min-consistency of the complete interval-valued fuzzy preference relation $\overline{R}^q = \left(\overline{r}_{ij}^q\right)_{n \times n}$, the following scaling conditions are used:

(i) If $r_{ij}^{q-} + r_{ji}^{q+} < 1$ and $r_{ij}^{q+} + r_{ji}^{q-} < 1$, then

$$\overline{r}_{ij}^q = \left[r_{ij}^{q-} + \frac{1 - (r_{ij}^{q-} + r_{ji}^{q+})}{2}, r_{ij}^{q+} + \frac{1 - (r_{ij}^{q+} + r_{ji}^{q-})}{2} \right] \tag{35}$$

and

$$\overline{r}_{ji}^q = \left[r_{ji}^{q-} + \frac{1 - (r_{ij}^{q+} + r_{ji}^{q-})}{2}, r_{ji}^{q+} + \frac{1 - (r_{ij}^{q-} + r_{ji}^{q+})}{2} \right]. \tag{36}$$

(ii) If $r_{ij}^{q-} + r_{ji}^{q+} < 1$ and $r_{ij}^{q+} + r_{ji}^{q-} > 1$, then

$$\overline{r}_{ij}^q = \left[r_{ij}^{q-} + \frac{1 - (r_{ij}^{q-} + r_{ji}^{q+})}{2}, r_{ij}^{q+} - \frac{r_{ij}^{q+} + r_{ji}^{q-} - 1}{2} \right] \tag{37}$$

and

$$\overline{r}_{ji}^q = \left[r_{ji}^{q-} - \frac{r_{ij}^{q+} + r_{ji}^{q-} - 1}{2}, r_{ji}^{q+} + \frac{1 - (r_{ij}^{q-} + r_{ji}^{q+})}{2} \right]. \tag{38}$$

(iii) If $r_{ij}^{q-}+r_{ji}^{q+}>1$ and $r_{ij}^{q+}+r_{ji}^{q-}<1$, then

$$\overline{r}_{ij}^{q}=\left[r_{ij}^{q-}-\frac{r_{ij}^{q-}+r_{ji}^{q+}-1}{2},r_{ij}^{q+}+\frac{1-(r_{ij}^{q+}+r_{ji}^{q-})}{2}\right] \tag{39}$$

and

$$\overline{r}_{ji}^{q}=\left[r_{ji}^{q-}+\frac{1-(r_{ij}^{q+}+r_{ji}^{q-})}{2},r_{ji}^{q+}-\frac{r_{ij}^{q-}+r_{ji}^{q+}-1}{2}\right]. \tag{40}$$

(iv) If $r_{ij}^{q-}+r_{ji}^{q+}>1$ and $r_{ij}^{q+}+r_{ji}^{q-}>1$, then

$$\overline{r}_{ij}^{q}=\left[r_{ij}^{q-}-\frac{r_{ij}^{q-}+r_{ji}^{q+}-1}{2},r_{ij}^{q+}-\frac{r_{ij}^{q+}+r_{ji}^{q-}-1}{2}\right] \tag{41}$$

and

$$\overline{r}_{ji}^{q}=\left[r_{ji}^{q-}-\frac{r_{ij}^{q+}+r_{ji}^{q-}-1}{2},r_{ji}^{q+}-\frac{r_{ij}^{q-}+r_{ji}^{q+}-1}{2}\right]. \tag{42}$$

A min-consistent matrix $\widetilde{\overline{R}}^{q}=\left(\widetilde{\overline{r}}_{ij}^{q}\right)_{n\times n}$ is obtained under these conditions.

Step 4: Determine the collective matrix $\overline{R}^{c}$against all experts, shown as follows:

$$\overline{R}^{c}=\left(\overline{r}_{ij}^{c}\right)_{n\times n}=\frac{1}{m}\left(\widetilde{\overline{r}}_{ij}^{1}+\widetilde{\overline{r}}_{ij}^{2}+\widetilde{\overline{r}}_{ij}^{3}...+\widetilde{\overline{r}}_{ij}^{m}\right)_{n\times n}, \tag{43}$$

where $1\leq i\leq n, 1\leq j\leq n$.
Step 5: Calculate the average degree $\overline{A}_i$ of alternative x_i over all other alternatives by using interval normalizing method:

$$\overline{A}_i=\frac{\sum_{j=1}^{n}\overline{r}_{ij}^{c}}{\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{r}_{ij}^{c}},\quad i=1,2,3,...,n. \tag{44}$$

Step 6: [21] Calculate the possibility degree $d_{ij}=d(\overline{A}_i\geq\overline{A}_j)$ by using the formula:

$$d(\overline{A}_i\geq\overline{A}_j)=\min\left\{\max\left(\frac{A_i^{+}-A_j^{-}}{A_i^{+}-A_j^{-}+A_j^{+}-A_i^{-}},0\right),1\right\} \tag{45}$$

and construct the complementry matrix $D=(d_{ij})_{n\times n}$, where $d_{ij}\geq 0$, $d_{ij}+d_{ji}=1$, $d_{ii}=0$, $i,j=1,2,3,...,n$.
Step 7: [33] Calculate the ranking value $R_V(x_i)$ of alternative x_i by using formula:

$$R_V(x_i)=\frac{2}{n^2}\sum_{j=1}^{n}d_{ij}, \tag{46}$$

where $1\leq i\leq n$ and $\sum_{i=1}^{n}R_V(x_i)=1$.

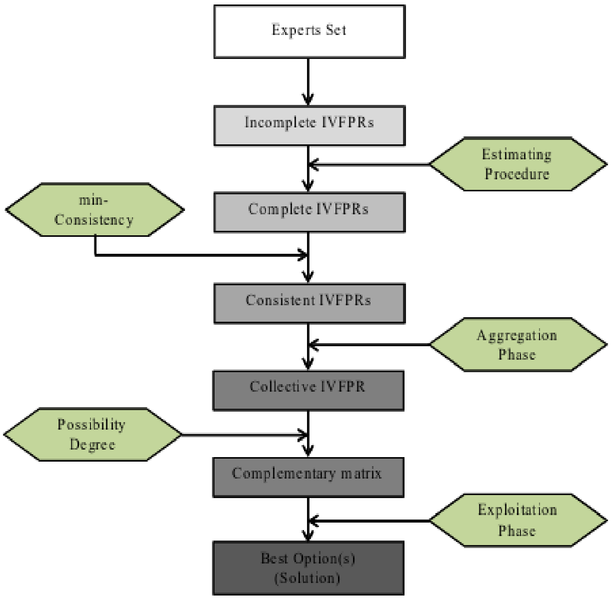

Figure 1: Resolution process for GDM with IVFPRs.

Example 3.1. A firm produces solar water refiners. In its production process, the company has to buy solar panels in different sizes and voltages from different suppliers. Presently, Japan Solar Company has four potential suppliers in four different countries, namely, Korea, China, Italy and Turkey, signified as $x_i(i = 1, 2, 3, 4)$, respectively. A committee consisting of three experts $E_q(q = 1, 2, 3)$ from different departments has been formed to assess the four suppliers $x_i(i = 1, 2, 3, 4)$. Suppose that the experts $E_q(q = 1, 2, 3)$ provide their assessments in the form of following incomplete IVFPRs:

$$\overline{R}^1 = \begin{bmatrix} [0.5, 0.5] & \overline{r}^1_{12} & [0.6, 0.8] & \overline{r}^1_{14} \\ \overline{r}^1_{21} & [0.5, 0.5] & \overline{r}^1_{23} & [0.3, 0.7] \\ [0.2, 0.4] & \overline{r}^1_{32} & [0.5, 0.5] & [0.6, 0.9] \\ \overline{r}^1_{41} & [0.3, 0.7] & [0.1, 0.4] & [0.5, 0.5] \end{bmatrix},$$

$$\overline{R}^2 = \begin{bmatrix} [0.5, 0.5] & \overline{r}_{12} & [0.4, 0.6] & [0.3, 0.7] \\ \overline{r}_{21} & [0.5, 0.5] & [0.7, 0.8] & \overline{r}_{24} \\ [0.4, 0.6] & [0.2, 0.3] & [0.5, 0.5] & [0.3, 0.4] \\ [0.3, 0.7] & \overline{r}_{42} & [0.6, 0.7] & [0.5, 0.5] \end{bmatrix},$$

and

$$\overline{R}^3 = \begin{bmatrix} [0.5, 0.5] & \overline{r}^3_{12} & [0.7, 0.8] & \overline{r}^3_{14} \\ [0.4, 0.6] & [0.5, 0.5] & \overline{r}^3_{23} & [0.5, 0.7] \\ [0.2, 0.3] & \overline{r}^3_{32} & [0.5, 0.5] & \overline{r}^3_{34} \\ \overline{r}^3_{41} & [0.3, 0.5] & \overline{r}^3_{43} & [0.5, 0.5] \end{bmatrix}.$$

Step 1: For the fuzzy preference relation $\overline{R}^1$, the sets of pairs of alternatives for known and unknown preference values are determined as follows:

$$\begin{aligned} K^1_P &= \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}, \\ U^1_P &= \{(1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (4, 1)\}. \end{aligned}$$

Step 2:

$$
\begin{aligned}
S_{12}^{11} &= \phi, \qquad S_{12}^{12} = \phi, \qquad S_{12}^{13} = \phi,\\
\overline{r}_{12}^{11} &= [0.5, 0.5], \qquad \overline{r}_{12}^{12} = [0.5, 0.5], \qquad \overline{r}_{12}^{13} = [0.5, 0.5],\\
\overline{r}_{12}^{1} &= \frac{1}{3}(\overline{r}_{12}^{11} + \overline{r}_{12}^{12} + \overline{r}_{12}^{13})\\
&= [0.5, 0.5].\\
K_P^{1'} &= \{(1,1),(1,2),(1,3),(2,2),(2,4),(3,1),(3,3),(3,4),(4,2),(4,3),\\
&\quad (4,4)\},\\
U_P^{1'} &= U_P^1 - \{(1,2)\} = \{(1,4),(2,1),(2,3),(3,2),(4,1)\}.\\
S_{14}^{11} &= \{2,3\}, \qquad S_{14}^{12} = \{3\}, \qquad S_{14}^{13} = \{2,3\},\\
\overline{r}_{14}^{11} &= \frac{1}{2}[T_{\min}(\overline{r}_{12}^{1}, \overline{r}_{24}^{1}) + T_{\min}(\overline{r}_{13}^{1}, \overline{r}_{34}^{1})] = \frac{1}{2}[[0.3, 0.5] + [0.6, 0.8]]\\
&= [0.45, 0.65],\\
\overline{r}_{14}^{12} &= T_{\min}([1,1] - \overline{r}_{13}^{1}, \overline{r}_{34}^{1}) = T_{\min}([0.2, 0, 4], [0.6, 0.9]) = [0.2, 0.4],\\
\overline{r}_{14}^{13} &= \frac{1}{2}[T_{\min}(\overline{r}_{12}^{1}, [1,1] - \overline{r}_{42}^{1}) + T_{\min}(\overline{r}_{13}^{1}, [1,1] - \overline{r}_{43}^{1})]\\
&= \frac{1}{2}[T_{\min}([0.5, 0.5], [0.3, 0.7]) + T_{\min}([0.6, 0.8], [0.6, 0.9])]\\
&= \frac{1}{2}[[0.3, 0.5] + [0.6, 0.8]] = [0.45, 0.65],\\
\overline{r}_{14}^{1} &= \frac{1}{3}(\overline{r}_{14}^{11} + \overline{r}_{14}^{12} + \overline{r}_{14}^{13}) = [0.3667, 0.5667].\\
K_P^{1''} &= \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,1),(3,3),(3,4),(4,2),\\
&\quad (4,3),(4,4)\},\\
U_P^{1''} &= U_P^{1'} - \{(1,4)\} = \{(2,1),(2,3),(3,2),(4,1)\}.
\end{aligned}
$$

Hence, continuing as above the fuzzy preference relation $\overline{R}^1$ against expert E_1 is obtained as follows:

$$
\overline{R}^1 = \begin{bmatrix}
[0.5, 0.5] & [0.5, 0.5] & [0.6, 0.8] & [0.3667, .5667]\\
[0.4667, 0.5222] & [0.5, 0.5] & [0.2889, 0.4574] & [0.3, 0.7]\\
[0.2, 0.4] & [0.3055, 0.5389] & [0.5, 0.5] & [0.6, 0.9]\\
[0.2519, 0.4673] & [0.3, 0.7] & [0.1, 0.4] & [0.5, 0.5]
\end{bmatrix}.
$$

Step 3: min-consistency preference relation $\overset{\sim}{\overline{R}}{}^1$ based on $\overline{R}^1$ is obtained as follows:

$$
\overset{\sim}{\overline{R}}{}^1 = \begin{bmatrix}
[0.5, 0.5] & [0.4889, 0.5166] & [0.6, 0.8] & [0.4497, .6574]\\
[0.4834, 0.5111] & [0.5, 0.5] & [0.375, 0.576] & [0.3, 0.7]\\
[0.2, 0.4] & [0.424, 0.625] & [0.5, 0.5] & [0.6, 0.9]\\
[0.3426, 0.5503] & [0.3, 0.7] & [0.1, 0.4] & [0.5, 0.5]
\end{bmatrix}.
$$

Likewise, min-consistency preference relations $\overset{\sim}{\overline{R}}{}^2$ and $\overset{\sim}{\overline{R}}{}^3$ against the experts E_2 and E_3 respectively, given as below:

$$\overline{\tilde{R}}^2 = \begin{bmatrix} [0.5, 0.5] & [0.33, 0.46] & [0.4, 0.6] & [0.3, 0.7] \\ [0.54, 0.67] & [0.5, 0.5] & [0.7, 0.8] & [0.46, 0.61] \\ [0.4, 0.6] & [0.2, 0.3] & [0.5, 0.5] & [0.3, 0.4] \\ [0.3, 0.7] & [0.39, 0.54] & [0.6, 0.7] & [0.5, 0.5] \end{bmatrix},$$

$$\overline{\tilde{R}}^3 = \begin{bmatrix} [0.5, 0.5] & [0.4166, 0.55] & [0.7, 0.8] & [0.4722, 0.599] \\ [0.45, 0.5834] & [0.5, 0.5] & [0.5333, 0.6592] & [0.5, 0.7] \\ [0.2, 0.3] & [0.3408, 0.4667] & [0.5, 0.5] & [0.384, 0.4994] \\ [0.401, 0.5278] & [0.3, 0.5] & [0.5006, 0.616] & [0.5, 0.5] \end{bmatrix}.$$

Step 4: The collective matrix against all the experts is shown as follows:

$$\overline{R}^c = \begin{bmatrix} [0.5, 0.5] & [0.4118, 0.5089] & [0.5667, 0.7333] & [0.4073, 0.6521] \\ [0.4911, 0.5882] & [0.5, 0.5] & [0.5361, 0.6784] & [0.42, 0.67] \\ [0.2667, 0.4333] & [0.3216, 0.4639] & [0.5, 0.5] & [0.428, 0.5998] \\ [0.3479, 0.5927] & [0.33, 0.58] & [0.4002, 0.572] & [0.5, 0.5] \end{bmatrix}$$

Step 5: The average degree $\overline{A}_i,\ i = 1, 2, 3, 4,$ of each alternative is derived by using interval normalizing method given as:

$$\overline{A}_1 = \frac{\sum_{j=1}^{4} \overline{r}^c_{1j}}{\sum_{i=1}^{4}\sum_{j=1}^{4} \overline{r}^c_{ij}} = \frac{[1.8858, 2.3943]}{[6.9274, 9.0726]} = [0.2078, 0.3456];$$

$$\overline{A}_2 = \frac{\sum_{j=1}^{4} \overline{r}^c_{2j}}{\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{r}^c_{ij}} = \frac{[1.9472, 2.4366]}{[6.9274, 9.0726]} = [0.2146, 0.3517];$$

$$\overline{A}_3 = \frac{\sum_{j=1}^{4} \overline{r}^c_{3j}}{\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{r}^c_{ij}} = \frac{[1.5163, 1.997]}{[6.9274, 9.0726]} = [0.1671, 0.2883];$$

$$\overline{A}_4 = \frac{\sum_{j=1}^{4} \overline{r}^c_{4j}}{\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{r}^c_{ij}} = \frac{[1.5781, 2.2447]}{[6.9274, 9.0726]} = [0.1739, 0.3240].$$

Step 6: By using eq. (45), the complementry matrix $D = (d_{ij})_{4\times 4}$ is obtained as follows:

$$D = (d_{ij})_{4\times 4} = \begin{bmatrix} 0.5 & 0.4765 & 0.6892 & 0.5964 \\ 0.5235 & 0.5 & 0.7147 & 0.619 \\ 0.3108 & 0.2853 & 0.5 & 0.4217 \\ 0.4036 & 0.381 & 0.5783 & 0.5 \end{bmatrix}$$

Step 7: The ranking value $R_v(x_i)$ of alternative $x_i,\ 1 \leq i \leq 4$, is obtained as follows:

$$
\begin{aligned}
R_v(x_1) &= \frac{2}{4^2}\sum_{j=1}^{4} d_{1j} = 0.2827625; \\
R_v(x_2) &= \frac{2}{4^2}\sum_{j=1}^{4} d_{2j} = 0.29465; \\
R_v(x_3) &= \frac{2}{4^2}\sum_{j=1}^{4} d_{3j} = 0.189725; \\
R_v(x_4) &= \frac{2}{4^2}\sum_{j=1}^{4} d_{4j} = 0.2328625;
\end{aligned}
$$

where $\sum_{i=1}^{4} R_v(x_i) = 1$. Thus, the final ranking of the alternatives is derived as follows:

$$x_2 > x_1 > x_4 > x_3.$$

Therefore, x_2 is the best alternative.

The numerical examples show the way to apply the proposed technique to construct the complete IVFPR based on min-consistency. In general, the proposed approach is quite easy for use in estimating unknown preference values.

4 Conclusion

In this paper the extended *minimum* t-norm has been used successfully to determine the missing values in incomplete IVFPR and further extends to construct the min-consistent matrix. Numerical studies show that the proposed technique can handle all type of incomplete IVFPR. Consequently, another algorithm is established to deal with GDM problems with incomplete IVFPRs. This process involves two stages, the estimation of unknown interval-valued preference values and the choice of the best alternative(s).

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A Fuzzy-based Decision Support Tool for Engineering Curriculum Design

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Abstract:
This paper describes a decision support tool which can be used for aiding the academic sraff in making the decision of including a specialty subject in an engineering curriculum. The approach is based on building a list of competences that should be acquired through the study of the specialty subjects. An evaluation of the competences is made by means of questionnaires and finally, a fuzzy model will be run. The output of the fuzzy model reflects the need for the evaluated specialty subject to be included in the curriculum. The proposed method takes into consideration the opinions and experience of both the academic staff and the employers.
Keywords: curriculum design, decision support engineering studies, fuzzy logic.

1 Introduction

National regulations state that engineering curriculum in the Romanian universities is composed mainly by four categories of subjects: fundamental subjects, domain subjects, specialty subjects and complementary subjects.

The fundamental subjects include Mathematics, Physics, Chemistry and other fundamental sciences which provide the future engineer the basic knowledge for operating with the laws and principles of engineering.

The domain subjects are intended to provide the technical knowledge related to a specific area of engineering, which at national level is called "domain". For example, some engineering domains in Romania are Mechanical Engineering, Industrial Engineering, Chemical Engineering, Electrical Engineering and many other.

The specialty subjects are intended to provide the specific knowledge which individualizes a study programme (also called "specialization") within and engineering domain. For example, Machine Building Technology and Machine-tools and Production Systems are two different specializations within the Industrial Engineering domain.

The complementary subjects are intended to provide the graduates the transversal competences and for engineering study programmes are usually selected from a list that include Foreign Language, History of Technics, Sports, Culture and Civilizations, Communication Techniques and so other.

The national authority responsible with quality assurance within higher education system in Romania, ARACIS (Romanian Agency of Quality Assurance in Higher Education) has established several rules regarding the distribution of fundamental, domain, specialty and complementary subjects within the curriculum. There are rules regarding the percentages of every subject category and there are also lists of subjects for every engineering domain from which the universities have to choose the subjects within the curriculum. However, while the lists of fundamental and domain disciplines are quite restrictive, for the specialty subjects, the constraints are more flexible and the list of possible "subject-candidates"/options is richer.

The broader range of options, which in fact may be considered beneficial, creates a set of problems and difficulties regarding the answer to the question: Which specialty subjects have/have not to be included in the curriculum? These problems have to be solved by each university, taking into consideration two aspects: the university autonomy, which encourage every school to customize its curriculum and the rapid evolution of the labor market, which have to be addressed by the universities by rapid adaptation of the curriculum.

The research presented in this paper was conducted in order to develop a tool for aiding the university staff in the decision process of designing the engineering curriculum with regards of the specialty subjects. The remaining part of this paper is as follows. In second section, a survey of the state of the art regarding various approaches regarding engineering curriculum design is presented. Third section is dedicated to building a competence list, considered necessary for an engineering graduate to possess. In the fourth section the flowchart of the proposed method is presented and the fifth section introduces the fuzzy model used for assessing the percent of introducing an evaluated subject in the curriculum. The last section is dedicated to the conclusion.

2 Previous work

Curriculum design for engineering studies should be oriented to final outcome of the higher education process, providing the labor market with a graduate able to fulfill the requirements of the employers [1]. The problem is that there is a certain conflict between education, which is seen as the main objective by the universities and professional training on the other hand, which is seen as mandatory requirement by the employers.

A literature survey shows that Competence-Based Learning [3, 4] is one of the concepts considered as base for higher education curriculum design in order to bridge the gap between education and professional training [2]. Competences are defined as a combination of skills and knowledge and it is considered that an individual's performance at work are hardly influenced by them [3]. Moreover, it is considered that competences comprises also entrepreneurial elements which are also needed by a higher education graduate [4, 5].

Consequently, university staff and other academic organizations involved in the curriculum design process have to define and select the required competences in order to define and structure a successful engineering study programme. In order to customize a specialization among other, within the same domain, it is very important to define the specific competences and to provide them by means of the specialty subjects, while keeping the curriculum as flexible as possible [6]. However, care has to be taken in order to avoid too much specialization or too much generalization [7].

Fuzzy logic was used quite often to build decision support tools related to educational process. Many works are related to applications of fuzzy logic in assessing the academic performance of the students [8–10]. The results obtained by using the method presented in [8] were compared to the values produced by statistical means. Another research, presented in [11], proposed a fuzzy logic approach to the assessment of student centered learning.

With the widespread of e-learning and web-based learning, special software products called Learning Management Systems (LMS) are used in order to provide the platform for learning environment. Fuzzy logic algorithms are reported to be used for the evaluation of LMS systems [12].

Competence-Based Learning had risen the problem of selection, analysis and evaluation the required competences for a higher education study programme. A literature survey revealed that fuzzy logic was used for this purpose, using a revised DACUM method (13). DACUM is the abbreviation for Developing A CurriculuUM and was invented by a group of scientists from New Start Company in Canada's Nova Scotia. By using fuzzy composite evaluation, the work presented in [13]developed a very effective method of analyzing technical competences for graduates of junior colleges.

An approach which uses both fuzzy logic and competence selection and analysis was presented in [14] and [15]. A particular fuzzy model for assisting the decision of including a specialty subject in the curriculum, for a specialization within Industrial Engineering domain. The approach was developed by a team from "Lucian Blaga" University of Sibiu, which includes the authors of the present paper.

3 The list of competences

The approach proposed in this work is intended to have high degree of generality, in order to be applied practically to any engineering study programme which can be related to an industrial process. For example, the method can be applied for machine-building and manufacturing engineers, hydraulic and pneumatic machines engineers, welding engineers, textile and leather engineers, chemical engineers, electrical engineers and many other engineering fields.

It is considered that in order to become a specialist in any of the above mentioned branch of engineering, the specialty subjects have to provide the student knowledge in five major components: technology, equipment, automation and control systems, CAD/CAE/CAM (computer aided design/engineering/manufacturing) and research and development, as shown in figure 1.

Consequently, a generalized list of competences which should be provided by the study of specialty subjects within the curriculum was built and presented below.

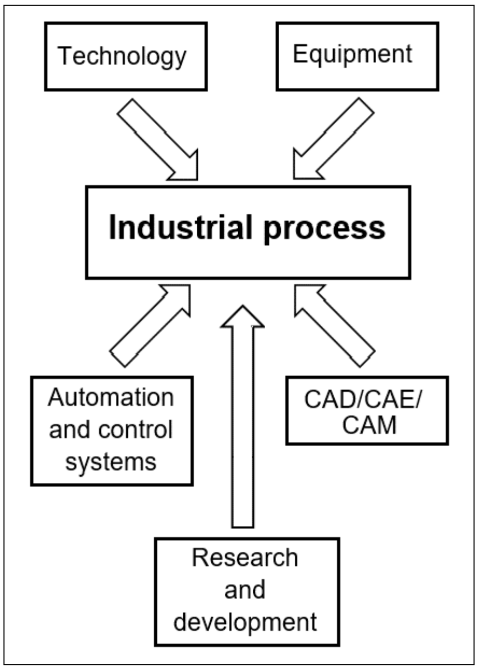

Figure 1: Major components of an industrial process

Competences related to **TECHNOLOGY** are grouped as follows:

- Competence to understand the basic principles of the *technology* (T1)

- Competence to implement the *technology* and to assembly the logistic chain for its implementation (T2)
- Competence to design the *technology* (T3)
- Competence to troubleshoot and optimize the *technology* (T4)

Competences related to *EQUIPMENT* are grouped as follows:

- Competence to understand the structure and operation principles of the *equipment* (E1)
- Competence to operate and maintain the *equipment* (E2)
- Competence to design the *equipment* (E3)
- Competence to troubleshoot and optimize the *equipment* (E4)

Competences related to *AUTOMATION AND CONTROL SYSTEMS* are grouped as follows:

- Competence to understand and apply the basic principles of automation and control systems for the *process* (A1)
- Competence to identify the components of the *process* which are suited for automation (A2)
- Competence to design, operate and maintain automation systems for the *process* (A3)
- Competence to troubleshoot and optimize automation systems for the *process* (A4)

Competences related to *CAD/CAE/CAM* are grouped as follows:

- Competence to operate with CAD/CAE/CAM software developed for the *process* (C1)
- Competence to build 3D models of single parts and assemblies within the structure of the *process* components (technology, equipment, automation systems) (C2)
- Competence to realize mathematical models and to simulate the behavior of the *process* components (C3) - Competence to manufacture *process* components by means of computer automated machining (C4)
- Competence to use methods and software for optimizing the *process* components (C5)

Competences related to *RESEARCH AND DEVELOPMENT* are grouped as follows:

- Competence to investigate and analyze new *processes, technologies, equipment and automation and control systems* (R1)
- Competence to perform fundamental research for the *process* (R2)
- Competence to build experimental layouts for applicative researches for the *process* (R3)
- Competence to produce and test prototype products for the *process* (R4)
- Competence to write technical reports about the *process* (R5)

In order to exemplify the high degree of generality, for every competences from the list the generalized item was italicized. It can be easily noticed that the italicized words can be replaced with more specific items, related to a specialty subject form the curriculum of an engineering specialization. For example, taking into consideration machine-building engineering as specialization and "Plastic deformation technologies and equipment" as specialty subject, the italicized items could be replaced by specific items as follows:

- *technology* → plastic deformation technology
- *equipment* → plastic deformation equipment
- *process* → plastic deformation process.

4 The flowchart of the proposed method

The flowchart of the proposed method is presented in figure 2.

The preliminary stage consists of two steps: developing a generalized set of competences and building a portfolio of specialty subjects.

The final stage consist of four steps. The first one involves the customization of the generalized set of competences for every evaluated subject, according to the specific requirements and characteristics of the study programme. As stated in the previous section, due to the high degree

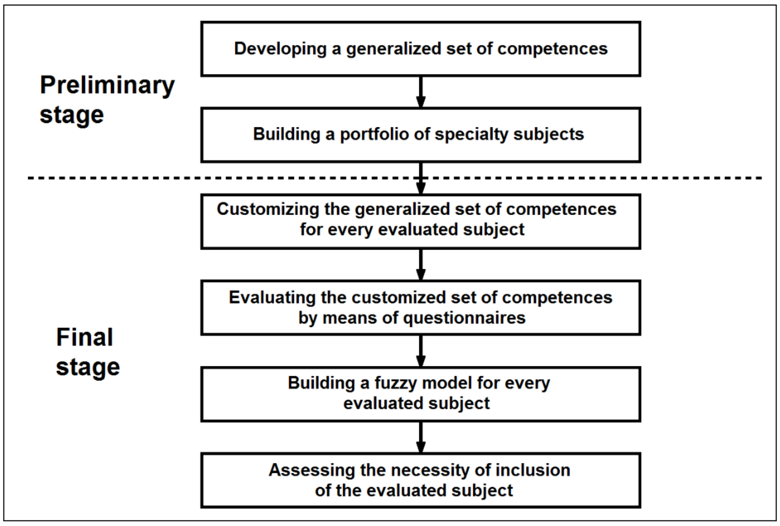

Figure 2: The flowchart of the proposed method

of generality of the generalized set of competences, the customization process is quite straightforward. The second step involves the evaluation of the customized set of competences by means of questionnaires. Because the method itself, the set of generalized and customized competences and the portfolio of specialty subjects are developed by the academic staff, the questionnaires are distributed to the other main stakeholders of the educational process, the employers.

The questionnaire (one for each evaluated subject) includes the set of customized competences, and asks the employers to grade each competence within the list with a grade from 1 to 10.

The next step involves the construction of a fuzzy model, also for every evaluated subject, which takes into consideration the results of the questionnaires for fuzzyfing the inputs.

Finally, after running the fuzzy model, the necessity of including the evaluated subject will be assessed.

5 The fuzzy model

The general structure of the fuzzy model is presented in figure 3. The inputs are the competences from the list presented in section 3.

For a certain specialty subject, the generalized fuzzy model has to be customized, by customizing the set of competences. Moreover, if the evaluated subject does not include one of the five categories of competences (for example a specialty subject with does not include technology-related chapters), that category can be removed from the inputs. Of course, for an engineering specialty subject it is hard to encounter such a situation, but nevertheless, it could appear.

The output variable of the fuzzy model is the "percentage of inclusion of the discipline (PI)" and will fluctuate as a percentage between 0 and 100%. Only the disciplines which obtain a percentage of inclusion equal or greater than a threshold (for example 80%) will be considered to be included in the curricula.

The linguistic variables used for the fuzzification of the inputs T1, T2, E1, E2, A1, A2, C1, C2, C3 and C4 were chosen as:useless;necessary; mandatory.

The linguistic variables used for the fuzzification of the inputs T3, T4, E3, E4, A3, A4, C5, R1, R2, R3, R4 and R5 were chosen as: inefficient; efficient.

The linguistic variables for the output were chosen as: low; medium; high.

The fuzzyfication of the inputs was done by using the membership functions presented in figure 4 (for T1, T2, E1, E2, A1, A2, C1, C2, C3 and C4) and figure 5 (for T3, T4, E3, E4, A3, A4, C5, R1, R2, R3, R4 and R5).

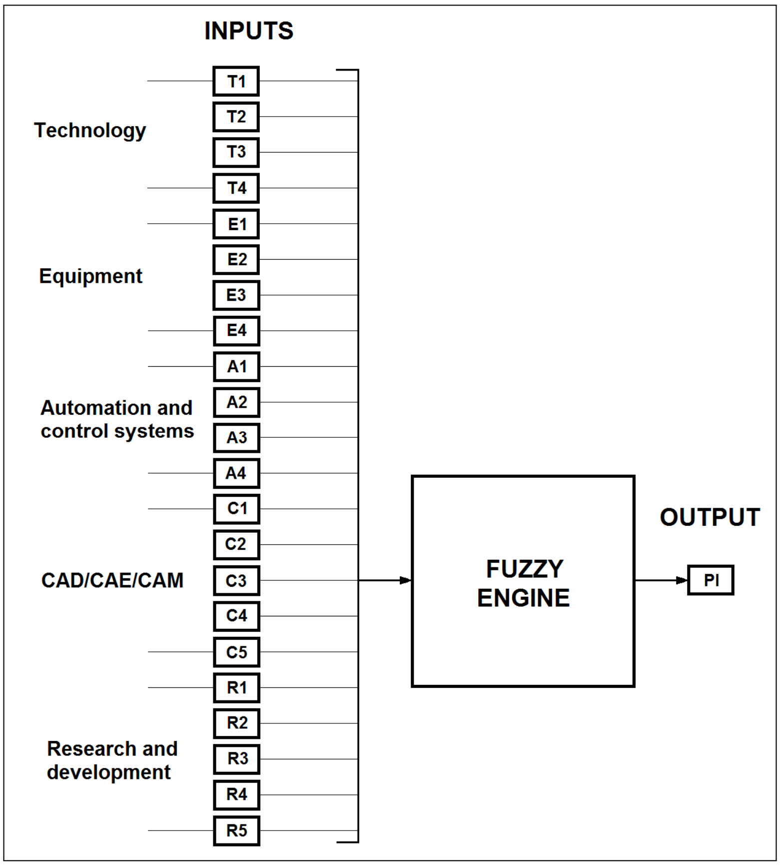

Figure 3: The structure for a fuzzy model

The membership function used for the fuzzification of the output is shown in figure 6.

The membership functions from figures 3-5 were built using only triangular and trapezoidal curves. The triangular curve is a function of a vector x and depends on three scalar parameters a, b and c as given by:

$$f(x,a,b,c) = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ \dfrac{c-x}{c-b}, & b \leq x \leq c \\ 0, & c \leq x \end{cases}$$

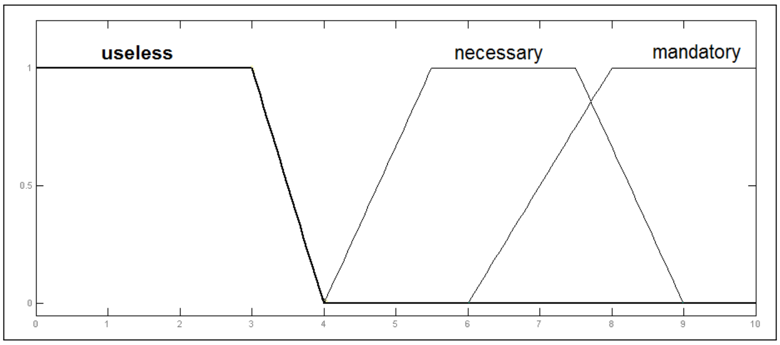

Figure 4: The membership function for fuzzyfication of the inputs T1, T2, E1, E2, A1, A2, C1, C2, C3 and C4

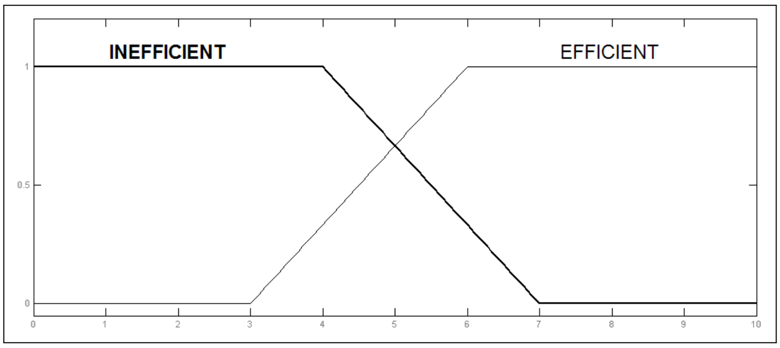

Figure 5: The membership function for fuzzyfication of the inputs T3, T4, E3, E4, A3, A4, C5, R1, R2, R3, R4 and R5

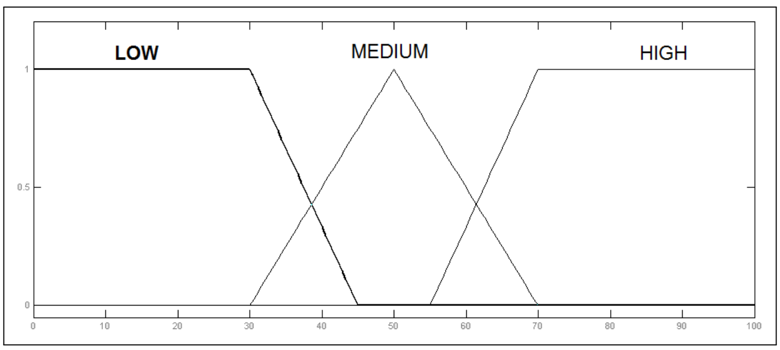

Figure 6: The membership function for fuzzyfication of the output

The trapezoidal curve is a function of a vector x and depends on four scalar parameters a, b, c and d as given by:

$$f(x,a,b,c,d) = \begin{cases} 0, & x \leq 0 \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b \leq x \leq c \\ \dfrac{d-x}{d-c}, & c \leq x \leq d \\ 0, & d \leq x \end{cases}$$

As development tool, the Matlab (with Fuzzy Logic Tcoolbox) was used. The other characteristics of the fuzzy model are: AND method minimum; OR method maximum; Implication method minimum; Aggregation method maximum; Defuzzification method centroid (center of gravity).

6 Conclusion

The main goal of this research was to develop a decision support tool, based on fuzzy logic for the decision of including or eliminating a specialty discipline within an engineering curriculum.

The proposed tool will help the academic staff to choose from a portfolio of specialty subjects. Usually, this portfolio is imposed by the national authorities for higher education in each country. Of course, the number of disciplines within the portfolio is higher as the number of disciplines which can be included in the curriculum, so the universities have to choose between them, a process which is often extremely difficult.

A list of competences which a future graduate of a four years engineering programme (seen as an industrial processes engineer) should possess after studying the subjects form above-mentioned portfolio was built. The competences within the list were divided into technology related, equipment related, automation and control systems related, CAD/CAM/CAE related and research and development related.

The list of competences were distributed (as questionnaires) to industrial companies in order to be assessed. The staff of companies, which are the main employer of the engineering graduates were asked to grade every competence from the list with grades between 1 and 10. Using the list of competences and the grades, fuzzy models were built for each subject within the portfolio. The inputs of each fuzzy model were the set of customized competences selected from the list and the output was the percent of inclusion in the curriculum. The fuzzyfication of the inputs was made according to a chosen set of membership functions and to the grades received by each competence.

Finally, the models were used to assist the decision of including the discipline in the curricula by allowing the user to calculate the percent of inclusion.

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Fuzzy Robust Tracking Control for Uncertain Nonlinear Time-Delay System

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Abstract: The problem of fuzzy robust tracking control is investigated for uncertain nonlinear time-delay systems. The nonlinear time-delay system is modeled as fuzzy Takagi-Sugeno (T-S) system, and fuzzy logic systems are used to eliminate the uncertainties of the system. A sufficient condition for the existence of fuzzy controller is given in terms of linear matrix inequalities (LMIs) and adaptive law. Based on Lyapunov stability theorem, the fuzzy control scheme guarantees the desired tracking performance in sense that all the closed-loop signals are uniformly ultimately bounded (UUB). Simulation results of 2-link manipulator demonstrate the effectiveness of the developed control scheme.
Keywords: fuzzy T-S model; fuzzy logic systems; nonlinear system; time-delay; tracking control.

1 Introduction

Fuzzy control approach offers a powerful and systematical control methodology to handle nonlinear system. Owing to the superior approximation and reasoning abilities of the fuzzy controller, fuzzy control approach has been applied in different applications. With the extensive efforts of the researchers working on the fuzzy control discipline, fruitful stability analysis results have been obtained to aid the design of stable fuzzy controllers. In [1], a fuzzy T-S model was employed to represent the system dynamics of the nonlinear system. The fuzzy T-S model represents the nonlinear system as a weighted sum of some linear subsystems. This particular structure offers a general framework to represent the nonlinear system which is favorable for system analysis. Fuzzy controllers [2-4] were proposed to handle the nonlinear system represented by the fuzzy T-S model. To avoid the effect of the uncertainties, a matching condition is assumed in [5–7], and an upper bound on uncertainties is introduced in [8–10]. The matching condition and the upper bound in dealing with the uncertainties are effective and feasible. However, there exists certain conservatism. The matching condition is a very conservative assumption and the upper bound may be too big or too small, which adds some difficulties to the controller design. On the other hand, it is well known that fuzzy logic systems can uniformly approximate nonlinear continuous functions to arbitrary accuracy. Thus, fuzzy logic systems are used to model uncertain nonlinear systems in [11–13].

Time delays are frequently encountered in engineering systems. The existence of time delays usually becomes the source of instability and degrading performance of systems. Therefore, stability analysis and controller synthesis for nonlinear time-delay systems are important both in theory and in practice.

By using fuzzy T-S model and fuzzy logic systems, we propose a novel robust tracking control scheme for a class of uncertain nonlinear time-delay system. Fuzzy T-S model is used to approximate the nonlinear system, and a fuzzy state feedback controller is designed to guarantee the stability of the fuzzy system. A compensator based fuzzy logic systems is introduced to eliminate the uncertainties of the system. The fuzzy control scheme ensures the desired tracking performance in sense that all the closed-loop signals are uniformly ultimately bounded (UUB).

The rest of the paper is organized as follows. Section 2 provides the problem formulation. Section 3 develops a procedure of the controller design. Section 4 gives the main result. Section 5 presents simulation examples to illustrate the effectiveness of the proposed method. These are followed by conclusions in Section 6.

2 Problem formulation

Consider the following uncertain nonlinear time-delay system:

$$\begin{aligned}
&\dot{x}_1 = x_2,\\
&\cdots\\
&\dot{x}_{(\beta_1-1)} = x_{\beta_1},\\
&\dot{x}_{\beta_1} = f_1(x, x(t-\tau_1), \cdots, x(t-\tau_r), u) + \tilde{f}_1(x, x(t-\tau_1), \cdots, x(t-\tau_r), u) + d_1,\\
&\dot{x}_{(\beta_1+1)} = x_{(\beta_1+2)},\\
&\cdots\\
&\dot{x}_n = f_m(x, x(t-\tau_1), \cdots, x(t-\tau_r), u) + \tilde{f}_m(x, x(t-\tau_1), \cdots, x(t-\tau_r), u) + d_m,
\end{aligned} \tag{1}$$

where $x = [x_1, \cdots, x_1^{(\beta_1-1)}, \cdots, x_{(n-\beta_m+1)}, \cdots, x_{(n-\beta_m+1)}^{(\beta_m-1)}]^T \in R^n$ with $\beta_1+\beta_2+\cdots+\beta_m = n$ and $u \in R^m$ are the system state and control input, respectively. f_i $(i = 1, \cdots, m)$ are known smooth nonlinear functions, $\tilde{f}_i$ $(i = 1, \cdots, m)$ are unknown nonlinear uncertainties, $\tau_i (i = 1, \cdots, r)$ are time delays, and d_i $(i = 1, \cdots, m)$ are external bounded disturbances.

The control objective of this paper is to find a fuzzy tracking controller such that, while maintaining all the closed-loop signals UUB, the system states of nonlinear system (1) follow those of the given stable reference model.

3 Fuzzy model, reference model and fuzzy controller

A fuzzy-model-based control system, formed by a fuzzy model, a reference model, and fuzzy controller connected in a closed-loop, is introduced.

3.1 Fuzzy model

A fuzzy dynamic model has been proposed by Takagi and Sugeno to represent a nonlinear system. The fuzzy dynamic model is described by the following fuzzy IF-THEN rules and will be employed here to deal with the control design problem for the nonlinear system in (1).

Plant Rule i: IF $z_1(t)$ *is* F_1^i *and,* $\cdots$ *, and* $z_s(t)$ *is* F_s^i*,THEN*

$$\dot{x}(t) = A_i x(t) + \sum_{l=1}^{r} A_{il} x(t-\tau_l) + B_i u(t) + d, \quad i = 1, \cdots, L \tag{2}$$

where $z_1(t), \cdots, z_s(t)$ are the premise variables, $F_j^i(j = 1, \cdots, s)$ are the fuzzy sets, L is the number of IF-THEN rules, A_i,B_i and A_{il} are some constant matrices with compatible dimensions, B_i=$[0, \cdots, b_{i1}^T, \cdots, 0, \cdots, b_{im}^T]^T \in R^{n\times m}$ with $b_{i1} \in R^m, \cdots, b_{im} \in R^m$, and $d = [0, \cdots, d_1, \cdots, 0, \cdots, d_m]^T$.

Then, the final output of the fuzzy system is inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{L} \mu_i [A_i x(t) + \sum_{l=1}^{r} A_{il} x(t-\tau_l)] + \sum_{i=1}^{L} \mu_i B_i u(t) + d, \tag{3}$$

where

$$\mu_i = v_i(z(t)) \Bigg/ \sum_{i=1}^{L} v_i(z(t)), v_i(z(t)) = \prod_{j=1}^{s} F_j^i(z_j(t)) \tag{4}$$

for all $t \geq 0$, and $F_j^i(z_j(t))$ is the grade of membership of $z_j(t)$ in F_j^i. It can be seen that $\sum\limits_{i=1}^{L} v_i(z(t)) > 0$, and $v_i \geq 0 (i = 1, \cdots, r)$ for all $t \geq 0$. We have $\mu_i \geq 0 (i = 1, \cdots, r)$, $\sum\limits_{i=1}^{L} \mu_i = 1$.

Hence, the nonlinear system (1) can be rearranged as the following equivalent system :

$$\dot{x}(t) = \sum_{i=1}^{L} \mu_i [A_i x(t) + \sum_{l=1}^{r} A_{il} x(t-\tau_l)] + \sum_{i=1}^{L} \mu_i B_i u(t) + B\Delta(x, x(t-\tau)) + d, \tag{5}$$

where $B\Delta(x, x(t-\tau)) = B\Delta(x, x(t-\tau_1), \cdots, x(t-\tau_r))$ denotes the uncertainties between the nonlinear system (1) and the fuzzy model (3), and $B = diag[B^1, \cdots, B^m]$ with $B^i = [0, \cdots, 0, 1]^T \in R^{\beta_i}$.

3.2 Reference model

The system states of nonlinear systems (1) are driven to follow those of the following stable reference model

$$\dot{x}_r(t) = A_r x_r(t) + r(t), \tag{6}$$

where $x_r(t)$ is a reference state, $r(t)$ is a bounded reference input, and A_r is an asymptotically stable matrix.

3.3 Fuzzy controller

A fuzzy controller is chosen as

$$u(t) = u_l(t) - u_f(t), \tag{7}$$

where $u_l(t)$ denotes the fuzzy state feedback control based on T-S model, and $u_f(t)$ is the adaptive compensator based on fuzzy logic systems. The former is used to stabilize the linear part of system (11), and the latter is used to compensate the uncertainties. $u_l(t)$ and $u_f(t)$ are designed as (8) and (10), respectively.

For the fuzzy model represented by (2) or (3), fuzzy state feedback control $u_l(t)$ shares the same IF parts with the following structure.

Control Rule i: IF $z_1(t)$ is F_1^i and, $\cdots$, and $z_s(t)$ is F_s^i , THEN

$$u_l(t) = K_i(x(t) - x_r(t)), \quad i = 1, \cdots, L.$$

Hence, the overall state feedback controller $u_l(t)$ is given by

$$u_l(t) = \sum_{i=1}^{L} \mu_i K_i(x(t) - x_r(t)), \tag{8}$$

where $K_i(i = 1, 2, \cdots, L)$ are matrices with proper dimensions and satisfy the following inequalities

$$\bar{A}_{ij}^T P + P\bar{A}_{ij} + \sum_{l=1}^{r} \alpha_l^{-1} P\bar{A}_{il}\bar{A}_{il}^T P + \sum_{l=1}^{r} \alpha_l I + \frac{1}{\rho^2} PP + \bar{Q} < 0, \quad i, j = 1, \cdots, L, \tag{9}$$

where $\bar{A}_{ij} = \begin{bmatrix} A_i + B_i K_j & -B_i K_j \\ 0 & A_r \end{bmatrix}$, $\bar{A}_{il} = \begin{bmatrix} A_{il} & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{Q} = diag\{2Q, 2Q\}$, P and Q are some symmetric and positive definite matrices, and $\alpha_l(l = 1, \cdots, r)$ are positive constants.

The adaptive compensator based on fuzzy logic systems $u_f(t)$ are as follows:

$$u_f(t) = \begin{cases} E^{-1}\hat{u}(x, x(t-\tau)|\Theta), \text{ if } E \text{ is nonsigular} \\ E^T(I + EE^T)^{-1}\hat{u}(x, x(t-\tau)|\Theta), \text{ if } E \text{ is sigular} \end{cases} \tag{10}$$

where $E_i = [b_{i1}^T, \cdots, b_{im}^T]^T \in R^{m\times m}$,$E = \sum\limits_{i=1}^{L} \mu_i E_i$, and $\hat{u}(x, x(t-\tau)|\Theta)$ is constructed by fuzzy logic systems. The weight Θ is an adaptive parameter, which is adapted by

$$\dot{\Theta} = \eta_1 \Psi^T(x, x(t-\tau))\bar{B}^T P\tilde{x}, \tag{11}$$

where η_1 is a positive constant, $\Psi(x, x(t-\tau))$ is a fuzzy basis-function matrix, and $\tilde{x} = [x^T, x_r^T]^T$.

In the following, we explain the solution of the inequalities (9) and the construction of fuzzy logic systems $\hat{u}(x, u|\Theta)$.

1) By Schur complements, the inequalities (9) are transformed into the LMIs. For the convenience of design, $P$is chosen as the form$P = diag\{P_1, P_2\}$, where P_1, P_2 are some symmetric and positive definite matrices. The inequalities (9) are equivalent to the following matrix inequalities

$$\begin{bmatrix} S_{11} & -P_1 B_i K_j & 0 \\ -(B_i K_j)^T P_1 & S_{22} & P_2 \\ 0 & P_2 & -\rho^2 I \end{bmatrix} < 0, \quad i, j = 1, 2, \cdots, L, \tag{12}$$

Where $S_{11} = P_1(A_i + B_i K_j) + (A_i + B_i K_j)^T P_1 + \sum\limits_{l=1}^{r} \alpha_l^{-1} P_1 A_{il} A_{il}^T P_1 + \sum\limits_{l=1}^{r} \alpha_l I + \frac{1}{\rho^2} P_1 P_1 + 2Q$,

$$S_{22} = P_2 A_r + A_r^T P_2 + \sum_{l=1}^{r} \alpha_l I + 2Q.$$

The matrix inequalities (12) imply $S_{11} < 0$. Let $W = P_1^{-1}$ and $Y_j = K_j W$. $S_{11} < 0$ is equivalent to the LMIs with prescribed Q and $\alpha_l(l = 1, \cdots, r)$,

$$\begin{bmatrix} S & W \\ W & -(\sum\limits_{l=1}^{r} \alpha_l I + 2Q)^{-1} \end{bmatrix} < 0, \quad i, j = 1, 2, \cdots, L \tag{13}$$

where $S = A_iW + WA_i^T + B_iY_j + (B_iY_j)^T + \sum_{l=1}^{r} \alpha_l^{-1} A_{il}A_{il}^T + (\rho^2)^{-1}I$.

By solving the LMIs (13), P_1 and $K_j(j = 1, 2, \cdots, L)$ could be obtained. And then, by substituting P_1 and $K_j(j = 1, 2, \cdots, L)$ into (12), (12) becomes standard LMIs. We can easily solve P_2 from (12). Therefore, the common solution P and $K_j(j = 1, 2, \cdots, L)$ could be found.

Remark 1: Either the matching condition or the upper bound is related to a large number of matrix operations. Without the matching condition and the upper bound, the dimension of the LMIs of this paper is reduced.

2) Fuzzy adaptive systems consist of four main components: fuzzy rule base, fuzzy inference engine, fuzzifier and defuzzifier [11]. The fuzzy rule base is composed of a collection of IF-THEN inference rules:

R^l: IF x_1 is $A_1^l, \cdots, x_n$ is A_n^lŁŹTHEN y is $G^l(l = 1, \cdots p)$
where $A_i^l(i = 1, \cdots, l)$ and $G^l(l = 1, \cdots p)$ are fuzzy sets. The kth element of $\Delta(x, x(t - \tau))$ is of the following form:

$$\hat{\Delta}_k(x, x(t - \tau)|\theta_k) = \xi_k^T(x, x(t - \tau))\theta_k,$$

where $\theta_k = (\theta_k^1, \cdots, \theta_k^p)^T \in R^p, \quad \xi_k^T(x, x(t - \tau)) = (\xi_k^1, \cdots, \xi_k^p) \in R^p$,

$$\xi_k^l = \prod_{i=1}^{n} \mu_{F_i^l}(x_i, x_i(t - \tau)) \Big/ \sum_{l=1}^{p} \prod_{i=1}^{n} \mu_{F_i^l}(x_i, x_i(t - \tau)), \mu_{F_i^l}(x_i, x_i(t-\tau)) = \mu_{F_i^l}(x_i) \prod_{j=1}^{r} \mu_{F_i^l}(x_i(t - \tau_j)),$$

and $\mu_{F_i^l}(x_i)(i = 1, 2, \cdots, n)$ are the membership functions.

In this paper, fuzzy logic systems are constructed to eliminate the uncertainties $\Delta(x, x(t-\tau))$. The approximation form is given as follows:

$$\hat{\Delta}(x, x(t - \tau)|\Theta) = \Psi(x, x(t - \tau))\Theta, \tag{14}$$

where $\Psi(x, x(t - \tau)) = diag[\xi_1^T(x, x(t - \tau)), \cdots, \xi_m^T(x, x(t - \tau))], \Theta = [\theta_1^T, \theta_2^T, \cdots, \theta_m^T]^T$.

Define the optimal the parameter Θ^* as

$$\Theta^* = \arg \min_{\Theta \in \Omega_1} [\sup_{x \in U_1} |\hat{u}(x, x(t - \tau)|\Theta) - \Delta(x, x(t - \tau))|], \tag{15}$$

where $U_1 = \{x \in R^n : \|x\| \le N\}$, $\Omega_1 = \{\Theta \in R^{pm}: \|\Theta\| \le \mathrm{M}\}$. U_1, Ω_1 denote the sets of suitable bounds on x,Θ respectively, $N, M$are upper bounds.

The approximation error for the function $\Delta(x, x(t - \tau))$can be expressed as

$$\hat{\Delta}(x, x(t - \tau)|\Theta) - \Delta(x, x(t - \tau)) = \Psi(x, x(t - \tau))\tilde{\Theta} + w, \tag{16}$$

where $\tilde{\Theta} = \Theta - \Theta^*$ the estimation error for Θ, $w = [w_1, \cdots, w_m]^T$is a residual term.
Remark 2: In order to guarantee $\|\Theta\| \le \mathrm{M}$, the adaptive law (11) must be modified by the projection algorithm [11] as follows:

$$\dot{\Theta} = \begin{cases} \eta_1 \Psi^T(x, x(t - \tau))\bar{B}^T P\tilde{x}, \text{if}(\|\Theta\| < \mathrm{M})\text{or}(\|\Theta\| = \mathrm{M} \text{ and } \tilde{x}^T \mathrm{P}\bar{B}\ \Psi(x, x(t - \tau))\Theta \le 0) \\ \mathrm{P}_\Theta[.], \text{if } \|\Theta\| = \mathrm{M} \text{ and } \tilde{x}^T \mathrm{P}\bar{B}\ \Psi(x, x(t - \tau))\Theta > 0 \end{cases}$$

where $\mathrm{P}_\Theta[.] = \eta_1 \Psi^T(x, x(t - \tau))\bar{B}^T P\tilde{x} - \eta_1 \frac{\tilde{x}^T \mathrm{P}\bar{B}\ \Psi(x, x(t-\tau))\Theta}{\|\Theta\|^2}$.

4 Stability analysis

Substituting (11) into (11) yields

$$\dot{x}(t) = \sum_{i=1}^{L}\mu_i[A_i x(t) + \sum_{l=1}^{r} A_{il}x(t-\tau_l)] + \sum_{i=1}^{L}\sum_{j=1}^{L}\mu_i\mu_j B_i K_j(x(t) - x_r(t))$$

$$-B(\hat{u}(x, x(t-\tau)|\Theta) - \Delta(x, x(t-\tau))) + d. \quad (17)$$

Let $\tilde{x}(t) = [x^T(t), x_r^T(t)]^T$, and$\bar{B} = [\ B^T \quad 0\ ]^T$. By using (11) and (17), a new extended closed-loop system is as follows:

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{L}\sum_{j=1}^{L}\mu_i\mu_j[\bar{A}_{ij}\tilde{x}(t) + \sum_{l=1}^{r}\bar{A}_{il}\tilde{x}(t-\tau_l)] + \bar{B}(-(\hat{u}(x, x(t-\tau)|\Theta) - \Delta(x, x(t-\tau))) + d\prime, \quad (18)$$

where $d\prime = [d^T, r^T(t)]^T$. When fuzzy logic systems $\hat{u}(x, x(t-\tau)|\Theta)$ could eliminate $\Delta(x, x(t-\tau))$, the closed-loop system (18) is stable.

By denoting $w\prime = [\bar{w}^T, r^T(t)]^T$, $\bar{w} = [0, \cdots, d_1 - w_1, \cdots, 0, \cdots, d_m - w_m]^T$ and using (14), the closed-loop system (18) could be rewritten as

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{L}\sum_{j=1}^{L}\mu_i\mu_j[\bar{A}_{ij}\tilde{x}(t) + \sum_{l=1}^{r}\bar{A}_{il}\tilde{x}(t-\tau_l)] + \bar{B}(-\Psi(x, x(t-\tau))\tilde{\Theta}) + w\prime. \quad (19)$$

From the above analysis, we have the following conclusion.

Theorem 1. *Given a matrix$Q > 0$, scalars$\rho > 0$,$\alpha_l(l = 1, \cdots, r) > 0$, $\eta_1 > 0$.If there exist matrices$P > 0$,$K_j(j = 1, 2, \cdots, L)$ such that the inequalities (9) hold. If the updating law for fuzzy logic systems is chosen as (11). Then there exists a controller (11) with the fuzzy state feedback controller (8) and the adaptive compensator (10) such that, while maintaining all the closed-loop signals UUB, the following tracking performance(20) is achieved*

$$\int_0^T (x(t) - x_r(t))^T Q(x(t) - x_r(t))dt \le \tilde{x}^T(0)P\tilde{x}(0) + \frac{1}{\eta_1}\tilde{\Theta}^T(0)\tilde{\Theta}(0) + \rho^2\int_0^T (w\prime^T w\prime)dt. \quad (20)$$

Proof: Consider the following Lyapunov-Krasoviskii candidate

$$V = \frac{1}{2}\tilde{x}^T P\tilde{x} + \frac{1}{2}\sum_{l=1}^{r}\int_{t-\tau_l}^{t}\alpha_l\tilde{x}^T(v)\tilde{x}(v)dv + \frac{1}{2\eta_1}\tilde{\Theta}^T\tilde{\Theta}, \quad (21)$$

where $\dot{V} = \dot{V}_1 + \dot{V}_2$, $\dot{V}_1$and $\dot{V}_2$ are given in (22) and (26), respectively.

$$\dot{V}_1 = \frac{1}{2}(\sum_{i=1}^{L}\sum_{j=1}^{L}\mu_i\mu_j[\bar{A}_{ij}\tilde{x}(t) + \sum_{l=1}^{r}\bar{A}_{il}\tilde{x}(t-\tau_l)])^T P\tilde{x}(t) + \frac{1}{2}\tilde{x}^T(t)P(\sum_{i=1}^{L}\sum_{j=1}^{L}\mu_i\mu_j[\bar{A}_{ij}\tilde{x}(t)$$

$$+\sum_{l=1}^{r}\bar{A}_{il}\tilde{x}(t-\tau_l)]) + \frac{1}{2}w\prime^T Px(t) + \frac{1}{2}x^T(t)Pw\prime + \frac{1}{2}\sum_{l=1}^{r}\alpha_l\tilde{x}^T(t)\tilde{x}(t) - \frac{1}{2}\sum_{l=1}^{r}\alpha_l\tilde{x}^T(t-\tau_l)\tilde{x}(t-\tau_l)$$

$$\le \frac{1}{2}(\sum_{i=1}^{L}\sum_{j=1}^{L}\mu_i\mu_j[\tilde{x}^T(t)\bar{A}_{ij}^T P\tilde{x}(t) + \tilde{x}^T(t)P\bar{A}_{ij}\tilde{x}(t) + \sum_{l=1}^{r}\alpha_l^{-1}\tilde{x}^T(t)P\bar{A}_{il}\bar{A}_{il}^T P\tilde{x}(t) + \sum_{l=1}^{r}\alpha_l\tilde{x}^T(t-\tau_l)\tilde{x}(t-\tau_l)]$$

$$-\frac{1}{2}(\frac{1}{\rho}Px(t)-\rho w\prime)^T(\frac{1}{\rho}Px(t)-\rho w\prime)+\frac{1}{2}\rho^2 w\prime^T w\prime+\frac{1}{2\rho^2}\tilde{x}^T(t)PP\tilde{x}(t)+\frac{1}{2}\sum_{l=1}^{r}\alpha_l\tilde{x}^T(t)\tilde{x}(t)$$

$$-\tfrac{1}{2}\sum_{l=1}^{r}\alpha_l\tilde{x}^T(t-\tau_l)\tilde{x}(t-\tau_l)$$

$$\leq\frac{1}{2}\sum_{i=1}^{L}\sum_{j=1}^{L}\mu_i\mu_j\tilde{x}^T(t)(\bar{A}_{ij}^TP+P\bar{A}_{ij}+\sum_{l=1}^{r}\alpha_l^{-1}P\bar{A}_{il}\bar{A}_{il}^TP+\sum_{l=1}^{r}\alpha_lI+\frac{1}{\rho^2}PP)\tilde{x}(t)+\frac{1}{2}\rho^2 w\prime^T w\prime. \quad (22)$$

Substituting (9) into (22) yields

$$\dot{V}_1\leq-\frac{1}{2}\tilde{x}^T(t)\bar{Q}\tilde{x}(t)+\frac{1}{2}\rho^2 w\prime^T w\prime. \quad (23)$$

By using (11),

$$V_2=[\tilde{x}^TP\bar{B}(-(\Psi(x,x(t-\tau))\tilde{\Theta})+\frac{1}{\eta_1}\tilde{\Theta}^T\dot{\Theta}]=0. \quad (24)$$

From (23)-(24),

$$\dot{V}\leq-\frac{1}{2}\tilde{x}^T(t)\bar{Q}\tilde{x}(t)+\frac{1}{2}\rho^2 w\prime^T w\prime. \quad (25)$$

When$\|\tilde{x}(t)\|>\frac{\rho}{\lambda_{\min}(\bar{Q})}\|w\prime\|$,$\dot{V}<0$.Thus, the closed-loop system consisting of (1), (11), (8) and (10) is UUB . □

Note that

$$\int_0^T(x(t)-x_r(t))^TQ(x(t)-x_r(t))dt \quad \leq\int_0^T\tilde{x}^T(t)\bar{Q}\tilde{x}(t)dt.$$

Integrating the above equation (25) from $t=0$ to Tyields (20).

5 Simulation example

In this section, we provide an example to verify the effectiveness of the proposed control scheme.

Example: Consider the following 2-link manipulator system in [14]

$$\ddot{q}(t)+C(q,\dot{q})\dot{q}(t)+g(q)=B(q)u(t)+\sum_{i=1}^{r}\xi_i(t)q(t-\tau_i)+d\prime, \quad (26)$$

where $C(q,\dot{q})=H^{-1}(q)C\prime(q,\dot{q}),g(q)=H^{-1}(q)g\prime(q),B(q)=H^{-1}(q)$,$d\prime=H^{-1}(q)d$,$q=[q_1,q_2]^T$, $\xi_i(t)(i=1,\cdots,r)$are uncertain and bounded, and $d$is the external bounded disturbance.

The reference model is as follows:

$$\dot{x}_r(t)=A_rx_r(t)+r(t), \quad (27)$$

where$A_r=diag\{A_{r1},A_{r2}\}$,$A_{r1}=A_{r2}=\begin{bmatrix}0&1\\-6&-5\end{bmatrix}$, $r(t)=[0,r_1(t),0,r_2(t)]^T$, $r_1(t)=r_2(t)=3\sin(2t)$.

Step1: Denote $x_1 = q_1, x_2 = \dot{q}_1, x_3 = q_2$, and $x_4 = \dot{q}_2$. Then, (26) can be written as a fourth-dimension system. A nine-rule fuzzy T-S model is used to approximate the nonlinear 2-link manipulator system at $x_1 = -\frac{\pi}{2}, 0, \frac{\pi}{2}$ and $x_3 = -\frac{\pi}{2}, 0, \frac{\pi}{2}$, where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5.927 & -0.001 & -0.315 & -0.0000084 \\ 0 & 0 & 0 & 1 \\ -6.859 & 0.002 & 3.155 & 0.0000062 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3.0428 & -0.0011 & -0.1791 & -0.0002 \\ 0 & 0 & 0 & 1 \\ -3.5436 & 0.0313 & 2.5611 & 0.0000114 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.2728 & 0.003 & 0.4339 & -0.0001 \\ 0 & 0 & 0 & 1 \\ -9.1041 & 0.0158 & -1.0574 & -0.000032 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.4535 & 0.0017 & 1.2427 & -0.0002 \\ 0 & 0 & 0 & 1 \\ -3.1873 & 0.0306 & -5.1911 & -0.000018 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 11.1336 & 0 & -1.8145 & 0 \\ 0 & 0 & 0 & 1 \\ -9.0918 & 0 & 9.1638 & 0 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1702 & -0.001 & 1.687 & -0.0002 \\ 0 & 0 & 0 & 1 \\ -2.3559 & 0.0314 & 4.5298 & -0.000011 \end{bmatrix},$$

$$A_7 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1206 & 0.0041 & 0.6205 & 0.0001 \\ 0 & 0 & 0 & 1 \\ 8.8794 & 0.0193 & -1.0119 & 0.000044 \end{bmatrix}, \quad A_8 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3.6421 & -0.0018 & 0.0721 & 0.0002 \\ 0 & 0 & 0 & 1 \\ 2.429 & -0.0305 & 2.9832 & -0.000019 \end{bmatrix},$$

$$A_9 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.2933 & -0.0009 & 0.2188 & -0.000012 \\ 0 & 0 & 0 & 1 \\ -7.4649 & 0.0024 & 3.2693 & -0.0000092 \end{bmatrix},$$

$$A_{11} = A_{21} = A_{31} = A_{41} = A_{51} = A_{61} = A_{71} = A_{81} = A_{91} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{12} = A_{22} = A_{32} = A_{42} = A_{52} = A_{62} = A_{72} = A_{82} = A_{92} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.01 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 2 \end{bmatrix}^T, B_2 = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T, B_3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix}^T,$$

$$B_4 = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T, B_5 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 2 \end{bmatrix}^T, B_6 = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T,$$

$$B_7 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix}^T, B_8 = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T, B_9 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 2 \end{bmatrix}^T.$$

The membership functions are adopted as the triangle type.

Step 2: On the basis of Theorem1, with$\alpha_1 = 0.005$, $\alpha_2 = 0.005$, and$\rho = 1$,we have

$$K_1 = \begin{bmatrix} \text{-76.9685} & \text{-42.9566} & \text{-19.6919} & \text{-8.9116} \\ 6.0025 & \text{-0.4619} & \text{-51.4252} & \text{-25.0336} \end{bmatrix},$$

$$K_2 = \begin{bmatrix} \text{-77.7828} & \text{-42.8754} & \text{-13.6211} & \text{-5.9413} \\ 8.7179 & 1.2251 & \text{-50.6614} & \text{-24.6859} \end{bmatrix},$$

$$K_3 = \begin{bmatrix} \text{-76.8347} & \text{-42.9089} & \text{-19.8204} & \text{-8.9785} \\ 5.8595 & \text{-0.5257} & \text{-51.3739} & \text{-25.0109} \end{bmatrix},$$

$$K_4 = \begin{bmatrix} \text{-77.7828} & \text{-42.8754} & \text{-13.6211} & \text{-5.9413} \\ 8.7179 & 1.2251 & \text{-50.6614} & \text{-24.6859} \end{bmatrix},$$

$$K_5 = \begin{bmatrix} \text{-77.7828} & \text{-42.8754} & \text{-13.6211} & \text{-5.9413} \\ 8.7179 & 1.2251 & \text{-50.6614} & \text{-24.6859} \end{bmatrix},$$

$$K_6 = \begin{bmatrix} \text{-77.7828} & \text{-42.8754} & \text{-13.6211} & \text{-5.9413} \\ 8.7179 & 1.2251 & \text{-50.6614} & \text{-24.6859} \end{bmatrix},$$

$$K_7 = \begin{bmatrix} \text{-79.8424} & \text{-43.4072} & \text{-6.0780} & \text{-2.2626} \\ 12.7745 & 3.6898 & \text{-50.2150} & \text{-24.4989} \end{bmatrix},$$

$$K_8 = \begin{bmatrix} \text{-77.7828} & \text{-42.8754} & \text{-13.6211} & \text{-5.9413} \\ 8.7179 & 1.2251 & \text{-50.6614} & \text{-24.6859} \end{bmatrix},$$

$$K_9 = \begin{bmatrix} \text{-80.1162} & \text{-43.5088} & \text{-5.8152} & \text{-2.1328} \\ 13.0509 & 3.8147 & \text{-50.3242} & \text{-24.5472} \end{bmatrix}.$$

Step 3: In fuzzy adaptive compensator, the membership functions are selected as

$$\mu_{F_i^1}(x_i) = \frac{1}{1+\exp[5(x_i+0.8)]}, \mu_{F_i^2}(x_i) = \exp[-(x_i+0.6)^2], \mu_{F_i^3}(x_i) = \exp[-(x_i+0.4)^2],$$

$$\mu_{F_i^4}(x_i) = \exp[-(x_i)^2], \mu_{F_i^5}(x_i) = \exp[-(x_i-0.4)^2], \mu_{F_i^6}(x_i) = \exp[-(x_i-0.6)^2],$$

$$\mu_{F_i^7}(x_i) = \frac{1}{1+\exp[5(x_i-0.8)]}, \quad i = 1, 2, \cdots, 4.$$

Step 4: Some parameters are choose as

$\eta_1 = 10, r = 2, \quad \tau_1 = 0.5, \tau_2 = 1, \xi_1(t) = 5 + 20\text{sin}(5\text{t})$, and$\xi_1(t) = 1 + 15\text{cos}(5\text{t})$,

$$\Theta(0) = [0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2],$$

$$(x_1(0), x_2(0), x_3(0), x_4(0), x_{r1}(0), x_{r2}(0), x_{r3}(0), x_{r4}(0)) = (0.4, 0, -0.4, 0, 0, 0, 0, 0).$$

By using the method in Theorem 1, the tracking performances of $x_1(t), x_2(t), x_3(t), x_4(t)$ are shown in Fig.1,and the control efforts $u_1(t)$ and $u_2(t)$ are given in Fig.2,respectively.

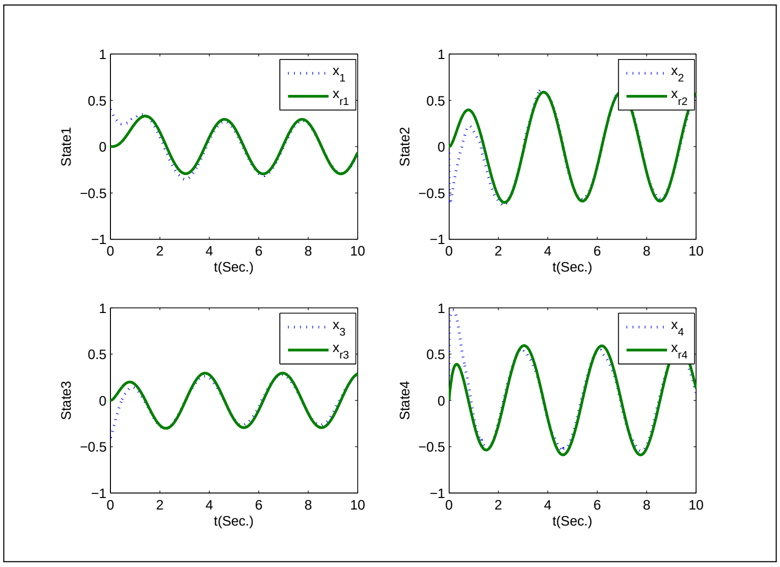

Figure 1: The responses of x_1,x_2,x_3,x_4, x_{r1},x_{r2},x_{r3}andx_{r4}

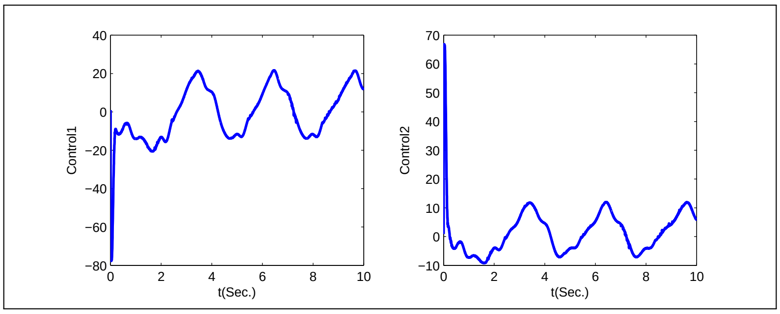

Figure 2: The control inputs u_1,u_2

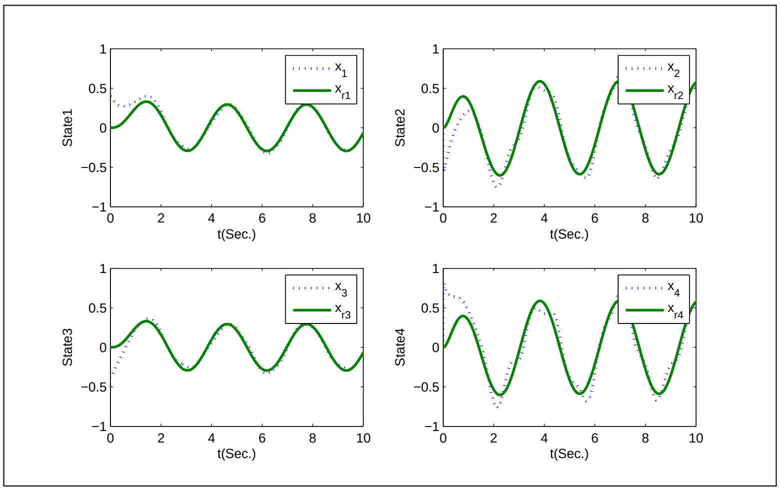

Figure 3: The responses of x_1,x_2,x_3,x_4, x_{r1},x_{r2},x_{r3}andx_{r4}

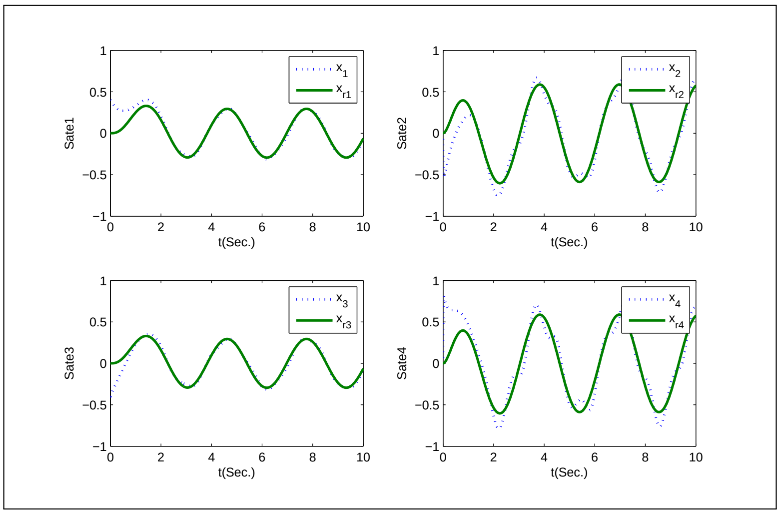

Figure 4: The responses of x_1,x_2,x_3,x_4, x_{r1},x_{r2},x_{r3}andx_{r4}

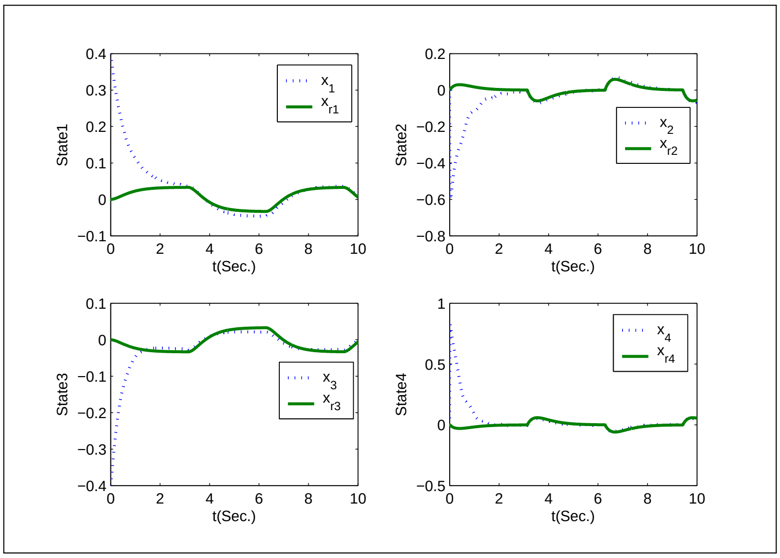

Figure 5: The responses of x_1,x_2,x_3,x_4, x_{r1},x_{r2},x_{r3}andx_{r4}

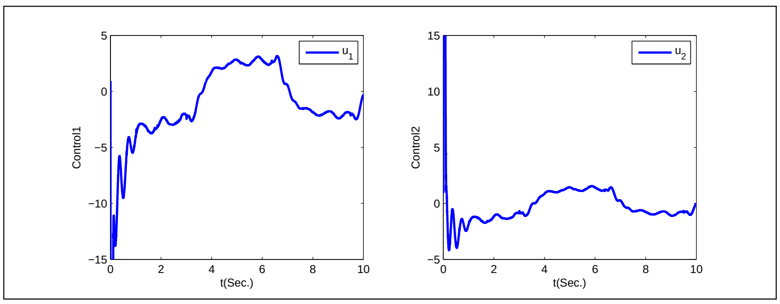

Figure 6: The control inputs u_1,u_2

When$\tau_1 = 1$,$\tau_2 = 1$, simulation results are shown in Fig.3.When$\tau_1 = 1$,$\tau_2 = 2$, simulation results are shown in Fig.4.

When $r_1(t)$and$r_2(t)$ are square waves having an amplitude ± 0.2 with a period of 2π, the tracking performances of $x_1(t), x_2(t), x_3(t), x_4(t)$ are shown in Fig. 5, and the control efforts $u_1(t)$ and $u_2(t)$ are given in Fig.6.

Simulation results illustrate that the proposed controller design is effective and feasible.

6 Conclusion

Based on fuzzy technique, a novel tracking control scheme is presented for uncertain nonlinear time-delay system. As main contribution of this paper, we design a novel fuzzy tracking controller, which is independent of the matching condition or the upper bound for the uncertainties. Furthermore, the tracking control design for discrete nonlinear systems is also developed.

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Why Fuzzy Cognitive Maps Are Efficient

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Abstract: In many practical situations, the relation between the experts' degrees of confidence in different related statements is well described by Fuzzy Cognitive Maps (FCM). This empirical success is somewhat puzzling, since from the mathematical viewpoint, each FCM relation corresponds to a simplified one-neuron neural network, and it is well known that to adequately describe relations, we need multiple neurons. In this paper, we show that the empirical success of FCM can be explained if we take into account that human's subjective opinions follow Miller's seven plus minus two law.
Keywords: fuzzy cognitive maps, neural networks, seven plus minus two law.

1 Introduction: Fuzzy Cognitive Maps and Their Puzzling Success

Need for (imprecise) expert estimates. To characterize a real-life system, we must know its properties. Some of these properties come from measurements and are thus represented by real numbers. However, in many cases, a large amount of information comes from expert estimates.

For example, to describe the current state of a patient, it is rarely enough to collect the corresponding measurement results – such a temperature, blood pressure, etc. Medical doctors supplement this information by providing imprecise ("fuzzy") estimates, such as "somewhat soft", "small", "rather high", etc.

Similarly, to adequately describe the financial situation of a company or of a country, it is important to supplement the corresponding numbers with expert estimates describing how probable is a default or, vice versa, how probable in an increase in profitability (and how big this increase can be). A typical expert's opinion sounds like this: "a vert big increase is improbable, but it is reasonable to expect a modest increase in reasonable time".

Fuzzy techniques as a natural way to describe imprecise expert estimates. When an expert completely agrees or completely disagrees with a precise statement (such as "the price of this stock will increase by at least 5% in a year"), in the computer, the resulting expert-estimated truth value of a statement is either "true" or "false". In the computer, "true" is usually represented as 1, and "false" as 0.

When the statement is imprecise, like the one above about a modest increase, the expert is not 100% sure that the price will increase by 5%. Instead, the expert has some degree of confidence in this 5% increase. Since full confidence in a statement is described by the number 1, and full confidence in its negation is described by the number 0, a reasonable way to describe the expert's partial confidence is by using numbers between 0 and 1: the higher the value, the larger the expert's degree of confidence. The use of numbers from the interval $[0, 1]$ for describing the experts' degrees of confidence is the main idea behind *fuzzy logic* [6, 9, 15].

From individual fuzzy properties to Fuzzy Cognitive Maps. Fuzzy properties describing a system are often interrelated, in the sense that some properties imply others. For example, in medicine, if a person is overweight and not very physically fit, this increases the possibility that this person may get diabetes and thus, may be in a pre-diabetic stage. In financial situations, if a company has many new patents, especially patents in a "hot" area like advanced bioinformatics, it is usually a good indication of its future financial prosperity, etc.

Fuzzy Cognitive Maps (FCM) are a way to describe the relations between different fuzzy properties. To describe these relations, for each property P, we first need to list all the properties $P_1, \ldots, P_n$ that directly affect the property P. Once this list is produced, we need to describe how the numerical values $x_1, \ldots, x_n \in [0, 1]$ of the properties P_i affect the value x of the property P. In computational terms, we need to come up with an algorithmic function $f(x_1, \ldots, x_n)$ that predicts the value x based on the known values $x_1, \ldots, x_n$.

Which functions should we choose? A natural idea is to start with the simplest functions. The simplest possible functions are linear functions, in which case we have

$$x = w_0 + w_1 \cdot x_1 + \ldots + w_n \cdot x_n. \tag{1}$$

However, we cannot simply use general linear functions:

- the predicted value should be within the interval $[0, 1]$, but
- for different combinations of weights, the above linear expression can be any real number, not necessarily a number between 0 and 1.

A reasonable idea is that, after we get the above linear combination, we then apply an additional transformation $s(x)$ that maps the whole real line to a number from the interval $[0, 1]$. In other words, instead of the linear expression (1), we use a slightly more complex expression

$$x = s(w_0 + w_1 \cdot x_1 + \ldots + w_n \cdot x_n), \tag{2}$$

where $s(x)$ is a pre-selected function that maps the real line into a unit interval $[0, 1]$. This function $s(x)$ is called an *activation function*.

This is the main idea behind *Fuzzy Cognitive Maps* (FCM); see, e.g., [2,3,7,8,11,12,16–20,22, 24–29]. The FCM model is used when experts provide estimates only for *some* of the properties. In this case, the values of other properties are estimates by using the corresponding formulas of type (2).

Which activations functions are used? Several different activation functions $s(x)$ have been used in FCM; the most frequently used is the *sigmoid function*

$$s(x) = \frac{1}{1 + \exp(-x)}. \tag{3}$$

The main reason why this function is used is the same reason why the same function is used in artificial neural networks: our goal is to describe human reasoning, and the sigmoid function provides a good approximate description of how similar processing in performed by the biological neurons in the brain; see, e.g., [1].

Comment. There are also theoretical reasons explaining why the sigmoid function is, in some reasonable sense, optimal; see, e.g., [10, 14, 23]. These theoretical reasons may also explain why evolution resulted in selecting this particular function in the actual brain – since this function is indeed optimal.

Fuzzy Cognitive Maps are efficient. In many practical applications, Fuzzy Cognitive Maps have led to a reasonably good description of human reasoning; see, e.g., [2, 3, 7, 8, 11, 12, 16–20, 22, 24–29].

This empirical success is puzzling. From the theoretical viewpoint, this empirical success is puzzling. Indeed, as we have mentioned, the output (3) of each corresponding fuzzy rule is the same as the output of a standard non-linear *neuron* [1, 24].

It is known that a 3-layer *neural network* has the *universal approximation property*; see, e.g., [1]. This means that if we use *several* (K) nonlinear neurons, with the outputs

$$x^{(k)} = s\left(w_0^{(k)} + w_1^{(k)} \cdot x_1 + \ldots + w_n^{(k)} \cdot x_n\right), \tag{4}$$

and then use an additional linear neuron to combine these output into a single combination

$$x = W^{(0)} + \sum_{k=1}^{K} W^{(k)} \cdot x^{(k)}, \tag{5}$$

then, for each continuous function $x = f(x_1, \ldots, x_n)$ on any box – in particular, on the box $[0,1] \times [0,1]$ – and for every $\varepsilon > 0$, we can find the values of the weights $w_i^{(k)}$ and $W^{(k)}$ for which, for every inputs, the final output (5) is ε-close to the desired value $f(x_1, \ldots, x_n)$.

It is also known that we need *several* neurons to get the universal approximation property, a single neuron does not have this property; see, e.g., [9]. And here, we have an opposite phenomenon: in many practical cases, already a single neuron provides a good approximation for the desired dependence! This is very puzzling.

Comment. The fact that a single neuron does not have a universal approximation property can be explained if we take into account that when the dependence $x = f(x_1, \ldots, x_n)$ is described by the formula (2), then for every i, we get

$$\frac{\partial f}{\partial x_i} = s\prime \cdot w_i,$$

where $s\prime(x)$ is the derivative of the activation function $s(x)$. Thus, for every $i \neq j$, we have

$$\frac{\partial f}{\partial x_i} = \text{const} \cdot \frac{\partial f}{\partial x_j},$$

where the constant is the ratio $\dfrac{w_i}{w_j}$. This property is already not satisfied by the simplest non-linear operation of multiplication $f(x_1, x_2) = x_1 \cdot x_2$, for which

$$\frac{\partial f}{\partial x_1} = x_2 \neq \text{const} \cdot \frac{\partial f}{\partial x_2} = \text{const} \cdot x_1.$$

What we do in this paper. In this paper, we provide a possible explanation for the (puzzling) empirical success of Fuzzy Cognitive Maps.

2 Possible Explanation for the Puzzling Empirical Success of FCMs

Main idea behind our explanation. Our main idea is to take into account the following difference between the general universal approximation property (as used in neural network theory) and what we want in Fuzzy Cognitive Maps.

The difference is that in the general applications of neural networks, the values $x_1, \ldots, x_n$, and x are usually well-defined physical quantities, quantities which can be, in principle, measured with an arbitrary accuracy ε. For example, if we use neural networks to design an appropriate control, we want the resulting control value x to be as close to the optimal value $f(x_1, \ldots, x_n)$ as possible.

In contrast, in Fuzzy Cognitive Maps, all the variables $x_1, \ldots, x_n$, and x are degrees of confidence describing expert opinions. These degrees are, by definition, imprecise, so computing them with too much for an accuracy simply does not make sense. An expert may be able to mark his or her degree of confidence by marking 6 on a scale from 0 to 10 – which corresponds to the degree of confidence 0.6 – but a normal expert cannot meaningfully distinguish between degrees of confidence 0.61 and 0.62.

Let us show that this difference can explain the puzzling empirical success of Fuzzy Cognitive Maps.

How accurate are expert estimates: 7 ± 2 rule. Psychologists have found out that we usually divide each quantity into 7 plus plus minus 2 categories – this is the largest number of categories whose meaning we can immediately grasp; see, e.g., [13, 21] (see also [30]). For some people, this "magical number" is $7 + 2 = 9$, for some it is $7 - 2 = 5$. This rule is in good accordance with the fact that in fuzzy logic, to describe the expert's opinion on each quantity, we usually use 7 ± 2 different categories (such as "small", "medium" , etc.).

Since on the interval $[0, 1]$, we can only have 7 ± 2 meaningfully different degrees of confidence, the accuracy of these degrees ranges from $1/9$ (for those who use 9 different categories) to $1/5$ (for those who use only 5 different categories).

What is the overall accuracy of the corresponding degrees. A Fuzzy Cognitive Map usually combines knowledge of a large number of experts. Since we have a large number of experts, it is practically certain that these experts include experts of all types: namely, those who can estimate their degree of confidence with a higher accuracy of $1/9$, as well as those who can only estimates their degree of confidence with a much lower relative accuracy of $1/5 = 20\%$.

In general, if we process a large amount of data of different accuracy, the accuracy of the result is determined by the lowest accuracy of the inputs. For example, if we estimate the overall amount of money $m = m_1 + m_2 + m_3$ owned by three people, and we know m_1 and m_2 with an accuracy of 1 cent, but we only know m_3 with an accuracy of 50% (i.e., we only know the ballpark estimate for m_3), then clearly our estimate for the sum m will be very inaccurate as well.

From this viewpoint, since FCM contains lower-accuracy data, with the accuracy 20%, we cannot expect the estimation results to be more accurate than that.

How accurate should our predictions be? Based on the above arguments, it makes sense to estimate the dependence of x on $x_1, \ldots, x_n$ with accuracy 20%. Attempts to get a more accurate estimation would be, in general, a useless computational exercise which is not related to the desired problem – of estimating the expert's degrees of confidence.

For example, if the expert's degree is 0.6, and our formula predicts 0.65, it is a very good match, and there is no need to come up with a formula that predicts exactly 0.6.

So, how many neurons do we need to make predictions with this accuracy: let us start our analysis. Let us show that in general, if we want predictions with accuracy 20%, then one neuron is sufficient.

Specifically, we will show that if, instead of taking the neuron that provides the largest contribution to the prediction, we consider both neurons, then – within the given accuracy – the result will not change.

It makes sense to treat the outputs of two neurons as random variables. As we have mentioned, for a general neural network, the result is a sum of the terms corresponding to different neurons. Let t_1 and t_2 be terms corresponding to the two neurons.

In general, these terms depends on many factors, so it makes sense to treat them as random variables.

As usual in statistics, we can somewhat simplify the problem by subtracting the means $E[t_i]$ from the corresponding variables. In precise terms, instead of the original random variables t_i, we can consider the differences $d_i \stackrel{\text{def}}{=} t_i - E[t_i]$ for which the mean value is 0: $E[d_i] = 0$.

What we compare. We compare the two situations:

- a situation in which we consider the sum $d_1 + d_2$ of both neural terms, and
- a situation in which we only have a single neuron, the one that provides the largest contribution:
  - we consider d_1 if $|d_1| \geq |d_2|$, and
  - we consider d_2 if $|d_2| \geq |d_1|$.

It is reasonable to assume that the variables corresponding to different neurons are independent. Since we have no reason to believe that the variables corresponding to different neurons are correlated, it makes sense to assume that the variables t_1 and t_2 – and thus, the corresponding differences d_1 and d_2 – are independent. This conclusion is in line with the general Maximum Entropy approach to dealing with probabilistic knowledge: if there are several possible probability distributions consistent with our knowledge, it makes sense to select the one which has the largest uncertainty (entropy; see, e.g., [4, 5]), i.e., to select a distribution for which the entropy

$$S = -\int \rho(x) \cdot \ln(\rho(x))\, dx$$

attains the largest possible value, where $\rho(x)$ is the probability density function (pdf).

In particular, for the case when for two random variables, we only know their marginal distributions, with probability densities $\rho_1(x_1)$ and $\rho_2(x_2)$, the Maximum Entropy approach selects the joint probability distribution with the probability density $\rho(x_1, x_2) = \rho_1(x_1) \cdot \rho_2(x_2)$ that corresponds exactly to the case when these two random variables are independent.

This independence make perfect sense for neural networks: when we train a neural network, we want to get a model which is as accurate as possible, and if we use two highly correlated neurons, we waste the second neuron to describe what the first neuron describes already.

How can we estimate the size of each random variable? For a random quantity with 0 mean, a natural measure of its size is its standard deviation σ.

If we only consider the term d_s corresponding to a single neuron, then this size can be described by the corresponding standard deviation σ_s. If we consider both neurons, then the size of the sum d_1+d_2 can be similarly characterized by its standard deviation σ_{12}. Since the variables are independent, the variance σ_{12}^2 of this sum is equal to the sum $\sigma_1^2 + \sigma_2^2$ of the corresponding variances. Thus, the standard deviation σ_{12} of the sum has the form

$$\sigma_{12} = \sqrt{\sigma_1^2 + \sigma_2^2}. \tag{6}$$

What we plan to prove. We plan to prove that the change caused by adding the second neuron is, in general, below the desired accuracy bound, i.e., that

$$\left|\frac{\sigma_{12}-\sigma_s}{\sigma_s}\right| \le 0.2. \tag{7}$$

Let us estimate the sizes σ_s and σ_{12} corresponding to two possible situations. We do not have much information about the size of the signals d_i corresponding to different neurons. We may guess some bounds $\underline{d} \le d_i \le \overline{d}$.

If all know about the probability distribution is that its values are always located on the interval $\left[\underline{d}, \overline{d}\right]$, then the Maximum Entropy approach recommends that we select a uniform distribution on this interval. This recommendation is in perfect accordance with common sense: if we have no reason to believe that some values from this interval are more probable or less probable then others, then it is reasonable to assume that all these values have the exact same probability, i.e., that the distribution is indeed unform.

For a uniform distribution on the interval $\left[\underline{d}, \overline{d}\right]$, the mean value is known to be equal to the midpoint $\dfrac{\underline{d}+\overline{d}}{2}$ of this interval. Since we are interested in random variables d_i with 0 mean, this means that this point must be equal to 0, i.e., that we have $\underline{d} = -\overline{d}$.

Since the mean is 0, the variance is equal to the expected value of d_i^2. Here, $d_i^2 = a_i^2$, where by $a_i \stackrel{\text{def}}{=} |d_i|$ we denoted the absolute value of the d_i. One can easily check that this absolute value a_i is uniformly distributed on the interval $[0, \overline{d}]$, with a constant probability density $\rho_i(x) = \dfrac{1}{\overline{d}}$, so its variance $\sigma^2 = \int x^2 \cdot \rho(x)\, dx$ is equal to

$$\sigma_i^2 = \int_0^{\overline{d}} x^2 \cdot \frac{1}{\overline{d}}\, dx = \frac{1}{3} \cdot \left(\overline{d}\right)^3 \cdot \frac{1}{\overline{d}} = \frac{1}{3} \cdot \left(\overline{d}\right)^2. \tag{8}$$

Due to the formula (6), we thus have

$$\sigma_{12} = \sqrt{\frac{1}{3} \cdot \left(\overline{d}\right)^2 + \frac{1}{3} \cdot \left(\overline{d}\right)^2} = \sqrt{\frac{2}{3}} \cdot \overline{d}. \tag{9}$$

Now, we need to estimate the variance σ_s^2 of the case when we only select one of the neurons, i.e., the expected value of the square of the selection d_s. Similarly to the one-neuron case, since $d_s^2 = |d_s|^2$, this variance is equal to the expected value of a_s^2, where we denoted $a_s \stackrel{\text{def}}{=} |d_s|$. By definition, $a_s = |d_s| = \max(|d_1|, |d_2|) = \max(a_1, a_2)$.

We know that a_1 and a_2 are two independent random variables which are uniformly distributed on the interval $\left[0, \overline{d}\right]$. The distribution of the maximum can be described in terms of the cumulative distribution functions (cdf) $F(x) \stackrel{\text{def}}{=} \text{Prob}(X \le x)$.

For the uniformly distributed variable a_1, we have $F_1(x) = \text{Prob}(a_1 \le x) = \dfrac{x}{\overline{d}}$. Similarly, $F_2(x) = \text{Prob}(a_2 \le x) = \dfrac{x}{\overline{d}}$. For the maximum $a_s = \max(a_1, a_2)$, we have

$$F_s(x) = \text{Prob}(a_s \le x) = \text{Prob}(\max(a_1, a_2) \le x).$$

Since the maximum of the two numbers is smaller than or equal to x if and only both of these numbers are $\le x$, we conclude that $F_s(x) = \text{Prob}((a_1 \le x)\ \&\ (a_2 \le x))$. The variables a_1 and

a_2 are independent, so

$$F_s(x) = \mathrm{Prob}(a_1 \le x) \cdot \mathrm{Prob}(a_2 \le x) = \frac{x}{\overline{d}} \cdot \frac{x}{\overline{d}} = \frac{x^2}{\left(\overline{d}\right)^2}. \tag{10}$$

From this cdf, we can compute the corresponding pdf $\rho_s(x)$:

$$\rho_s(x) = \frac{dF_s(x)}{dx} = \frac{2x}{\left(\overline{d}\right)^2}. \tag{11}$$

Thus, the desired variance σ_s^2 has the form

$$\sigma_s^2 = \int_0^{\overline{d}} x^2 \cdot \frac{2x}{\left(\overline{d}\right)^2}\,dx = \frac{2}{\left(\overline{d}\right)^2} \cdot \int_0^{\overline{d}} x^3\,dx = \frac{2}{\left(\overline{d}\right)^2} \cdot \frac{1}{4} \cdot \left(\overline{d}\right)^4 = \frac{1}{2} \cdot \left(\overline{d}\right)^2. \tag{12}$$

Thus,

$$\sigma_s = \sqrt{\frac{1}{2}} \cdot \overline{d}. \tag{13}$$

Final step: checking that the desired inequality (7) is indeed satisfied. Now that we have expressions (9) and (13) for the sizes σ_{12} and σ_s, we can plug them into the inequality (7) and check that this inequality is satisfied – i.e., that within the desired accuracy of 20%, adding the second neuron, on average, does not matter. Indeed, substituting expressions (9) and (13) into the left-hand side of the formula (7) and dividing both the numerator and the denominator by the common factor $\overline{d}$, we get the ratio

$$r = \left| \frac{\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{2}}} \right|.$$

Dividing both terms in the numerator by the denominator, we get

$$r = \left| \sqrt{\frac{4}{3}} - 1 \right| = \left| \frac{2}{\sqrt{3}} - 1 \right| = \left| \frac{2}{3} \cdot \sqrt{3} - 1 \right|.$$

For $\sqrt{3} = 1.73\ldots$, we get

$$r = \left| \frac{2 \cdot 1.73\ldots}{3} - 1 \right| = \left| \frac{3.46\ldots}{3} - 1 \right| = |1.15\ldots - 1| = 0.15\ldots < 0.2.$$

The statement is proven.

3 Conclusion

Thus, we have explained why Fuzzy Cognitive Maps (i.e., 1-neuron neural networks) are adequate for describing the dependence between the experts' degree of confidence, when the relative accuracy of 20% is quite sufficient.

Comment.

We have proven that, *on average*, the relative error does not exceed 20%. This explains why Fuzzy Cognitive Maps are efficient in many practical situations. However, the fact that this inequality is satisfied on average does not necessarily mean that it is satisfied always. There may be cases when Fuzzy Cognitive Maps do not work that well – in this case, it makes sense to describe the corresponding dependencies $x = f(x_1, \ldots, x_n)$ by generic (multi-neuron) neural networks.

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Fuzzy Continuous Mappings in Fuzzy Normed Linear Spaces

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Abstract: In this paper we continue the study of fuzzy continuous mappings in fuzzy normed linear spaces initiated by T. Bag and S.K. Samanta, as well as by I. Sadeqi and F.S. Kia, in a more general settings. Firstly, we introduce the notion of uniformly fuzzy continuous mapping and we establish the uniform continuity theorem in fuzzy settings. Furthermore, the concept of fuzzy Lipschitzian mapping is introduced and a fuzzy version for Banach's contraction principle is obtained. Finally, a special attention is given to various characterizations of fuzzy continuous linear operators. Based on our results, classical principles of functional analysis (such as the uniform boundedness principle, the open mapping theorem and the closed graph theorem) can be extended in a more general fuzzy context.
Keywords: Fuzzy normed linear spaces; fuzzy continuous mapping; fuzzy bounded linear operators.

1 Introduction and preliminaries

The concept of fuzzy set was introduced by L. Zadeh [14] in 1965. If X is a nonempty set, a fuzzy set in X is a function μ from X into the unit interval $[0,1]$. The classical union and intersection of ordinary subsets of X can be extended by the following formulas, proposed by L. Zadeh

$$\left(\bigvee_{i\in I}\mu_i\right)(x)=\sup\{\mu_i(x)\ :\ i\in I\}\ ,\ \left(\bigwedge_{i\in I}\mu_i\right)(x)=\inf\{\mu_i(x)\ :\ i\in I\}\ .$$

From here to the notion of fuzzy topological space, there was one more step to be taken. Thus, in 1968, C.L. Chang [4] introduced the notion of fuzzy topological space. The definition is a natural translation to fuzzy sets of the ordinary definition of topological space. Indeed, a fuzzy topology is a family $\mathcal{T}$, of fuzzy sets in X, such that $\mathcal{T}$ is closed with respect to arbitrary union and finite intersection and every constant function belong to $\mathcal{T}$.

One of the important problems concerning the fuzzy topological spaces is to obtain an adequate notion of fuzzy metric space. Many authors have investigated this question and several notions of fuzzy metric space have been defined and studied. We just mention the definition given by I. Kramosil and J. Michálek [9] in 1975.

Definition 1. The pair (X,M) is said to be a fuzzy metric space if X is an arbitrary set and M is a fuzzy set in $X\times X\times[0,\infty)$ satisfying the following conditions:

(M1) $M(x,y,0)=0,(\forall)x,y\in X;$

(M2) $(\forall)x,y\in X, x=y$ if and only if $M(x,y,t)=1$ for all $t>0$;

(M3) $M(x,y,t)=M(y,x,t),(\forall)x,y\in X,(\forall)t>0;$

(M4) $M(x,z,t+s)\geq M(x,y,t)*M(y,z,s)$, $(\forall)x,y,z\in X,(\forall)t,s>0;$

(M5) $(\forall)x, y \in X, M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous and $\lim_{t\to\infty} M(x, y, t) = 1$.

We note that, in previous definition, $*$ denotes a continuous t-norm (see [13]). The basic examples of continuous t-norms are $\wedge, \cdot, *_L$, which are defined by $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ (usual multiplication in $[0, 1]$) and $a *_L b = \max\{a + b - 1, 0\}$ (the Lukasiewicz t-norm).

In studying fuzzy topological linear spaces, A.K. Katsaras [8], in 1984, first introduced the notion of fuzzy norm on a linear space. Since then many mathematicians have introduced several notions of fuzzy norm from different points of view. Thus, C. Felbin [6] in 1992 introduced an idea of fuzzy norm on a linear space by assigning a fuzzy real number to each element of linear space. In 1994, S.C. Cheng and J.N. Mordeson [5] introduced a concept of fuzzy norm on a linear space whose associated metric is Kramosil and Michálek type. Following S.C. Cheng and J.N. Mordeson, in 2003, T. Bag and S.K. Samanta [2] proposed another concept of fuzzy norm.

In this paper we continue the study of fuzzy continuous mappings in fuzzy normed linear spaces initiated by T. Bag and S.K. Samanta [3], as well as by I. Sadeqi and F.S. Kia [12], in a more general settings:

Definition 2. [10] Let X be a vector space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and $*$ be a continuous t-norm. A fuzzy set N in $X \times [0, \infty)$ is called a fuzzy norm on X if it satisfies:

(N1) $N(x, 0) = 0, (\forall)x \in X$;

(N2) $[N(x, t) = 1, (\forall)t > 0]$ if and only if $x = 0$;

(N3) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right), (\forall)x \in X, (\forall)t \geq 0, (\forall)\lambda \in \mathbb{K}^*$;

(N4) $N(x + y, t + s) \geq N(x, t) * N(y, s), (\forall)x, y \in X, (\forall)t, s \geq 0$;

(N5) $(\forall)x \in X$, $N(x, \cdot)$ is left continuous and $\lim_{t\to\infty} N(x, t) = 1$.

The triple $(X, N, *)$ will be called fuzzy normed linear space (briefly FNLS).

Remark 3. a) T. Bag and S.K. Samanta [2], [3] gave a similar definition for $* = \wedge$, but in order to obtain some important results they assumed that the fuzzy norm also satisfied the following conditions:

(N6) $N(x, t) > 0, (\forall)t > 0 \Rightarrow x = 0$;

(N7) $(\forall)x \neq 0, N(x, \cdot)$ is a continuous function and strictly increasing on the subset $\{t : 0 < N(x, t) < 1\}$ of $\mathbb{R}$.

The results obtained by T. Bag and S.K. Samanta [3], as well as by I. Sadeqi and F.S. Kia [12], can be found in this more general setting.
b) I. Goleţ [7], C. Alegre and S. Romaguera [1] also gave this definition in the context of real vector spaces.
c) $N(x, \cdot)$ is nondecreasing, $(\forall)x \in X$.

Example 4. *[2] Let X be a linear space and $||\cdot||$ be a norm on X. Let*

$$N(x, t) := \begin{cases} 1 & if \quad |x| < t \\ 0 & if \quad |x| \geq t \end{cases}$$

Then $(X, N, \wedge)$ is a FNLS. In particular, $(\mathbb{C}, N, \wedge)$ is a FNLS.

Theorem 5. *[10] Let $(X, N, *)$ be a FNLS. For $x \in X, r \in (0,1), t > 0$ we define the open ball*

$$B(x, r, t) := \{y \in X \ : \ N(x - y, t) > r\} \ .$$

Then

$$\mathcal{T}_N := \{T \subset X \ : \ x \in T \text{ iff } (\exists)t > 0, r \in (0,1) \ : \ B(x, r, t) \subseteq T\}$$

is a topology on X.

Moreover, if the t-norm $*$ satisfies $\sup\limits_{x \in (0,1)} x * x = 1$, then $(X, \mathcal{T}_N)$ is Hausdorff.

Theorem 6. *[10] Let $(X, N, *)$ be a FNLS. Then $(X, \mathcal{T}_N)$ is a metrizable topological vector space.*

Definition 7. [2] Let $(X, N, *)$ be a FNLS and (x_n) be a sequence in X.

1. The sequence (x_n) is said to be convergent if $(\exists)x \in X$ such that

$$\lim_{n \to \infty} N(x_n - x, t) = 1 \ , \ (\forall)t > 0 \ .$$

   In this case, x is called the limit of the sequence (x_n) and we denote $\lim\limits_{n \to \infty} x_n = x$ or $x_n \to x$.

2. The sequence (x_n) is called Cauchy sequence if

$$\lim_{n \to \infty} N(x_{n+p} - x_n, t) = 1 \ , \ (\forall)t > 0, (\forall)p \in \mathbb{N}^* \ .$$

3. $(X, N, *)$ is said to be complete if any Cauchy sequence in X is convergent to a point in X. A complete FNLS will be called a fuzzy Banach space.

Theorem 8. *Let $(X, N, *)$ be a FNLS and*

$$p_\alpha(x) := \inf\{t > 0 \ : \ N(x, t) > \alpha\}, \alpha \in (0,1) \ .$$

Then, for $x \in X, s > 0, \alpha \in (0,1)$, we have:

$$p_\alpha(x) < s \text{ if and only if } N(x, s) > \alpha \ .$$

Proof: The proof is entirely the same as in [10], where there are considered FNLSs of type $(X, N, \wedge)$. □

The structure of the paper is as follows: in Section 2, we introduce the notion of uniformly fuzzy continuous mapping and we establish the uniform continuity theorem in fuzzy settings. The concept of fuzzy Lipschitzian mapping is introduced and a fuzzy version for Banach's contraction principle is obtained. In Section 3, special attention is given to various characterizations of fuzzy continuous linear operators. Based on our results, classical principles of functional analysis (such as the uniform boundedness principle, the open mapping theorem and the closed graph theorem) can be extended in a more general fuzzy context.

Even if the structure of fuzzy F-spaces, recently introduced in [11], is much more complicated than that of fuzzy Banach spaces, we intent to study, in a further paper, fuzzy continuous linear operators on fuzzy F-spaces and to prove that the well-known principles of functional analysis are valid in this context too.

In the following sections $(X, N_1, *_1)$, $(Y, N_2, *_2)$ will be FNLSs with the t-norms $*_1, *_2$ which satisfy $\sup\limits_{x \in (0,1)} x *_i x = 1, (\forall)i = 1, 2$.

2 Fuzzy continuous mappings

Definition 9. [3] A mapping $T : X \to Y$ is said to be fuzzy continuous at $x_0 \in X$, if

$$(\forall)\varepsilon > 0, (\forall)\alpha \in (0,1), (\exists)\delta = \delta(\varepsilon, \alpha, x_0) > 0, (\exists)\beta = \beta(\varepsilon, \alpha, x_0) \in (0,1)$$

$$\text{such that } (\forall)x \in X : N_1(x - x_0, \delta) > \beta \text{ we have that } N_2(T(x) - T(x_0), \varepsilon) > \alpha \;.$$

If T is fuzzy continuous at each point of X, then T is called fuzzy continuous on X.

Theorem 10. *[3] A mapping $T : X \to Y$ is fuzzy continuous at $x_0 \in X$, if and only if $(\forall)(x_n) \subseteq X$, $x_n \to x_0$, we have that $T(x_n) \to T(x_0)$.*

Definition 11. A mapping $T : X \to Y$ is said to be uniformly fuzzy continuous on X, if

$$(\forall)\varepsilon > 0, (\forall)\alpha \in (0,1), (\exists)\delta = \delta(\varepsilon, \alpha) > 0, (\exists)\beta = \beta(\varepsilon, \alpha) \in (0,1)$$

$$\text{such that } (\forall)x, y \in X : N_1(x - y, \delta) > \beta \text{ we have that } N_2(T(x) - T(y), \varepsilon) > \alpha \;.$$

Remark 12. If T is uniformly fuzzy continuous, then T is fuzzy continuous.

Theorem 13. (Uniform continuity theorem). *Let $(X, N_1, *_1)$ be a compact FNLS and $(Y, N_2, *_2)$ be a FNLS. If $T : X \to Y$ is a fuzzy continuous mapping, then T is uniformly fuzzy continuous.*

Proof: Let $\varepsilon > 0$ and $\alpha \in (0,1)$.

As $\sup\limits_{x \in (0,1)} x *_2 x = 1$, then there exists $\alpha_0 \in (0,1)$ such that $\alpha_0 *_2 \alpha_0 > \alpha$.

As $T : X \to Y$ is a fuzzy continuous on X, for all $x \in X$, there exist $\delta_x = \delta\left(\frac{\varepsilon}{2}, \alpha_0, x\right) > 0$, $\beta_x = \beta\left(\frac{\varepsilon}{2}, \alpha_0, x\right) \in (0,1)$ such that

$$(\forall)y \in X \;:\; N_1(x - y, \delta_x) > \beta_x \Rightarrow N_2\left(T(x) - T(y), \frac{\varepsilon}{2}\right) > \alpha_0 \;.$$

As $\sup\limits_{x \in (0,1)} x *_1 x = 1$, we can take $\gamma_x > \beta_x$ such that $\gamma_x *_1 \gamma_x > \beta_x$.

Since X is compact and $\left\{B\left(x, \gamma_x, \frac{\delta_x}{2}\right)\right\}_{x \in X}$ is an open covering of X, there exist $x_1, x_2, \cdots, x_n$ in X such that $X = \bigcup\limits_{i=1}^{n} B\left(x_i, \gamma_{x_i}, \frac{\delta_{x_i}}{2}\right)$. Let $\beta = \max\{\gamma_{x_i}\}$ and $\delta = \min\left\{\frac{\delta_{x_i}}{2}\right\}$, for $i = 1, 2, \cdots, n$.

Let $x, y \in X$ arbitrary, such that $N_1(x - y, \delta) > \beta$. As $x \in X$, there exists $i \in \{1, 2, \cdots, n\}$ such that $x \in B\left(x_i, \gamma_{x_i}, \frac{\delta_{x_i}}{2}\right)$, namely $N_1\left(x - x_i, \frac{\delta_{x_i}}{2}\right) > \gamma_{x_i}$. Hence

$$N_1(x - x_i, \delta_{x_i}) \geq N_1\left(x - x_i, \frac{\delta_{x_i}}{2}\right) > \gamma_{x_i} > \beta_{x_i} \;.$$

Thus

$$N_2\left(T(x) - T(x_i), \frac{\varepsilon}{2}\right) > \alpha_0 \;.$$

We remark that

$$N_1(y - x_i, \delta_{x_i}) \geq N_1\left(y - x, \frac{\delta_{x_i}}{2}\right) *_1 N_1\left(x - x_i, \frac{\delta_{x_i}}{2}\right) \geq$$

$$\geq N_1(y - x, \delta) *_1 N_1\left(x - x_i, \frac{\delta_{x_i}}{2}\right) > \beta *_1 \gamma_{x_i} \geq \gamma_{x_i} *_1 \gamma_{x_i} > \beta_{x_i} \;.$$

Thus $N_2\left(T(y)-T(x_i),\frac{\varepsilon}{2}\right)>\alpha_0$.

In conclusion

$$N_2(T(x)-T(y),\varepsilon)\geq N_2\left(T(x)-T(x_i),\frac{\varepsilon}{2}\right)*_2 N_2\left(T(x_i)-T(y),\frac{\varepsilon}{2}\right)>$$

$$>\alpha_0 *_2 \alpha_0>\alpha\ .$$

□

Definition 14. A mapping $T:X\to Y$ is said to be fuzzy Lipschitzian on X if $(\exists)L>0$ such that

$$N_2(T(x)-T(y),t)\geq N_1\left(x-y,\frac{t}{L}\right)\ ,\ (\forall)t>0,(\forall)x,y\in X\ .$$

If $L<1$ we say that T is a fuzzy contraction.

Remark 15. It is clear that a fuzzy Lipschitzian mapping is necessarily fuzzy continuous.

Theorem 16. (Banach's contraction principle). *Let $(X,N,*)$ be a fuzzy Banach space and $T:X\to X$ be a fuzzy contraction. Then T has a unique fixed point $z\in X$ and*

$$\lim_{n\to\infty}T^n(x)=z,(\forall)x\in X\ .$$

Proof: Let $x\in X$ be arbitrary. Then $\{T^n(x)\}$ is a Cauchy sequence. Indeed, for $t>0$ and $p\in\mathbb{N}^*$, we have

$$N(T^{n+p}(x)-T^n(x),t)\geq N\left(T^{n+p-1}(x)-T^{n-1}(x),\frac{t}{L}\right)\geq$$

$$\geq\cdots\geq N\left(T^p(x)-x,\frac{t}{L^n}\right).$$

As $L\in(0,1)$, we have that $\lim\limits_{n\to\infty}\frac{t}{L^n}=\infty$. Thus

$$\lim_{n\to\infty}N\left(T^p(x)-x,\frac{t}{L^n}\right)=1\ .$$

Hence $\lim\limits_{n\to\infty}N(T^{n+p}(x)-T^n(x),t)=1$, namely $\{T^n(x)\}$ is a Cauchy sequence.

Since X is complete, we have that $\{T^n(x)\}$ is a convergent sequence. Thus $(\exists)z\in X$ such that $\lim\limits_{n\to\infty}T^n(x)=z$. We note that

$$z=\lim_{n\to\infty}T^{n+1}(x)=\lim_{n\to\infty}T(T^n(x))=T(z)\ .$$

Now we show the uniqueness. Suppose that there exist $z,y\in X,z\neq y$ with the property $z=T(z),y=T(y)$. As $z\neq y$, there exists $s>0$ such that $N(z-y,s)=a<1$. Then, for all $n\in\mathbb{N}$, we have

$$a=N(y-z,s)=N(T^n(y)-T^n(z),s)\geq N\left(y-z,\frac{s}{L^n}\right)\to 1\ .$$

Thus $a=1$, which contradicts our assumption. □

3 Fuzzy continuous linear operators

Theorem 17. *Let $T : X \to Y$ be a linear operator. Then T is fuzzy continuous on X, if and only if T is fuzzy continuous at a point $x_0 \in X$.*

Proof: " $\Rightarrow$ " It is obvious.
" $\Leftarrow$ " Let $y \in Y$ be arbitrary. We will show that T is fuzzy continuous at y. Let $\varepsilon > 0, \alpha \in (0,1)$. Since T is fuzzy continuous at $x_0 \in X$, there exist $\delta > 0$, $\beta \in (0,1)$ such that

$$(\forall)x \in X : N_1(x - x_0, \delta) > \beta \Rightarrow N_2(T(x) - T(x_0), \epsilon) > \alpha \ .$$

Replacing x by $x + x_0 - y$, we obtain that

$$(\forall)x \in X : N_1(x + x_0 - y - x_0, \delta) > \beta \Rightarrow N_2(T(x + x_0 - y) - T(x_0), \epsilon) > \alpha \ ,$$

namely

$$(\forall)x \in X : N_1(x - y, \delta) > \beta \Rightarrow N_2(T(x) - T(y), \epsilon) > \alpha \ .$$

Thus T is fuzzy continuous at $y \in Y$. As y is arbitrary, it follows that T is fuzzy continuous on $D(T)$. □

Corollary 18. *Let $T : X \to Y$ be a linear operator. Then T is fuzzy continuous on X, if and only if*

$$(\forall)\varepsilon > 0, (\forall)\alpha \in (0,1), (\exists)\delta = \delta(\epsilon, \alpha) > 0, (\exists)\beta = \beta(\epsilon, \alpha) \in (0,1) \textit{such that}$$
$$(\forall)x \in X : N_1(x, \delta) > \beta \textit{ we have that } N_2(T(x), \varepsilon) > \alpha \ .$$

Theorem 19. *A linear operator $T : X \to Y$ is fuzzy continuous on X, if and only if $(\forall)\alpha \in (0,1), (\exists)\beta = \beta(\alpha) \in (0,1), (\exists)M = M(\alpha) > 0$ such that*

$$(\forall)t > 0, (\forall)x \in X \ : \ N_1(x, t) > \beta \Rightarrow N_2(T(x), Mt) > \alpha \ .$$

Proof: " $\Leftarrow$ " Let $\varepsilon > 0$, $\alpha \in (0,1)$ be arbitrary. Then there exist $\beta = \beta(\alpha) \in (0,1)$, $M = M(\alpha) > 0$ such that

$$(\forall)t > 0, (\forall)x \in X \ : \ N_1(x, t) > \beta \Rightarrow N_2(T(x), Mt) > \alpha \ .$$

In particular, for $t = \frac{\varepsilon}{M}$, we obtain

$$N_1\left(x, \frac{\varepsilon}{M}\right) > \beta \Rightarrow N_2(T(x), \varepsilon) > \alpha \ .$$

Applying Corollary 18, for $\delta = \frac{\varepsilon}{M} > 0$, we obtain that T is fuzzy continuous on X.
" $\Rightarrow$ " We suppose that $(\exists)\alpha_0 \in (0,1)$ such that

$$(\forall)\beta \in (0,1), (\forall)M > 0, (\exists)t_0 = t_0(\beta, M) > 0, (\exists)x_0 = x_0(\beta, M) \in X,$$

$$N_1(x_0, t_0) > \beta \text{ and } N_2(T(x), Mt_0) \leq \alpha_0 \ .$$

The set $V_0 = \{y \in Y \ : \ N_2(y, t_0) > \alpha_0\}$ is an open neighborhood of 0_Y. We will prove that, for all neighborhood U of 0_X, we have $T(U) \not\subseteq V_0$, which contradicts the fuzzy continuity of T at 0_X. As $\{B(0, \beta, s)\}_{\beta \in (0,1), s>0}$ is a fundamental system of neighborhoods of 0_X, it is enough to show that for all $\beta \in (0,1), s > 0$ we have $T(B(0, \beta, s)) \not\subseteq V_0$.

As $M > 0$ is arbitrary, we can chose $s = \frac{t_0}{M}$. We note that, for $z_0 = \frac{1}{M}x_0 \in X$, we have

$$N_1\left(z_0, \frac{t_0}{M}\right) = N_1\left(\frac{1}{M}x_0, \frac{t_0}{M}\right) = N_1(x_0, t_0) > \beta \ .$$

Hence $z_0 \in B\left(0, \beta, \frac{t_0}{M}\right)$. We will prove that $T(z_0) \notin V_0$, namely $N_2(T(z_0), t_0) \leq \alpha_0$. Indeed,

$$N_2(T(z_0), t_0) = N_2\left(T\left(\frac{1}{M}x_0\right), t_0\right) = N_2(T(x_0), Mt_0) \leq \alpha_0 \ .$$

□

Corollary 20. *A linear functional $f : (X, N_1, *) \to (\mathbb{C}, N, \wedge)$ is fuzzy continuous, if and only if $(\exists)\beta \in (0,1), (\exists)M > 0$ such that*

$$(\forall)t > 0, (\forall)x \in X, N_1(x,t) > \beta \Rightarrow |f(x)| < Mt \ .$$

Proof: According to the previous theorem f is fuzzy continuous if and only if

$$(\forall)\alpha \in (0,1), (\exists)\beta \in (0,1), (\exists)M > 0 \text{ such that}$$

$$(\forall)t > 0, (\forall)x \in X \ : \ N_1(x,t) > \beta \Rightarrow N(f(x), Mt) > \alpha \ .$$

But

$$N(f(x), Mt) > \alpha \Leftrightarrow N(f(x), Mt) = 1 \Leftrightarrow |f(x)| < Mt \ .$$

Hence $(\exists)\beta \in (0,1), (\exists)M > 0$ such that

$$(\forall)t > 0, (\forall)x \in X, N_1(x,t) > \beta \Rightarrow |f(x)| < Mt \ .$$

□

Corollary 21. *Let $(X, N_1, *_1)$, $(Y, N_2, *_2)$ be FNLSs and*

$$p_\alpha(x) := \inf\{t > 0 \ : \ N_1(x,t) > \alpha\}, \alpha \in (0,1) \ ,$$

$$q_\alpha(x) := \inf\{t > 0 \ : \ N_2(x,t) > \alpha\}, \alpha \in (0,1) \ .$$

A linear operator $T : X \to Y$ is fuzzy continuous on X if and only if

$$(\forall)\alpha \in (0,1), (\exists)\beta = \beta(\alpha) \in (0,1), (\exists)M = M(\alpha) > 0$$

$$\textit{such that}\; q_\alpha(Tx) \leq Mp_\beta(x) \ , \ (\forall)x \in X \ .$$

Proof: According to the previous theorem,

$$(\forall)\alpha \in (0,1), (\exists)\beta = \beta(\alpha) \in (0,1), (\exists)M = M(\alpha) > 0 \text{ such that}$$

$$(\forall)t > 0, (\forall)x \in X \ : \ N_1(x,t) > \beta \Rightarrow N_2(T(x), Mt) > \alpha \ .$$

Thus, for $x \in X$, we have

$$\{t > 0 \ : \ N_1(x,t) > \beta\} \subseteq \{t > 0 \ : \ N_2(Tx, Mt) > \alpha\} \ .$$

Hence

$$\inf\{t > 0 \ : \ N_1(x,t) > \beta\} \geq \inf\{t > 0 \ : \ N_2(Tx, Mt) > \alpha\} \ ,$$

namely $\inf\{t > 0 \ : \ N_1(x,t) > \beta\} \geq \inf\left\{\frac{t}{M} > 0 \ : \ N_2(Tx,t) > \alpha\right\}$. Therefore

$$p_\beta(x) \geq \frac{1}{M}q_\alpha(Tx), \forall)x \in X \ .$$

□

Corollary 22. *A linear functional* $f : (X, N_1, *) \to (\mathbb{C}, N, \wedge)$ *is fuzzy continuous, if and only if* $(\exists)\beta \in (0,1), (\exists)M > 0$ *such that*

$$|f(x)| \leq Mp_\beta(x), (\forall)x \in X \; .$$

Remark 23. We note that a subset A of a topological linear space X is said to be bounded if for every neighbourhood V of 0_X, there exists a positive number k such that $A \subset kV$. A linear operator $T : X \to Y$ is said to be bounded if T maps bounded sets into bounded sets. Based on this remark the following definitions are natural.

Definition 24. [12] A subset A of X is called fuzzy bounded, if $(\forall)\alpha \in (0,1)$, $(\exists)t_\alpha > 0$ such that $A \subset B(0, \alpha, t_\alpha)$.

Definition 25. [12] A linear operator $T : X \to Y$ is said to be fuzzy bounded if T maps fuzzy bounded sets of X into fuzzy bounded sets of Y.

We must note that the following result was established by I. Sadeqi and F.S. Kia [12] for FNLSs of type $(X, N, \wedge)$ which satisfy $(N7)$. Since the proof is entirely the same as in [12], it is omitted.

Theorem 26. *Let* $T : X \to Y$ *be a linear operator. The following sentences are equivalent:*

1. *T is fuzzy continuous;*
2. *T is topological continuous;*
3. *T is fuzzy bounded;*

4 Conclusion

As fuzzy continuity and topological continuity are equivalent and since FNLSs are metrizable topological linear spaces, all results and theorems in topological linear spaces hold for FNLSs. Particularly, we can obtain fuzzy versions for the classical principles of functional analysis (such as the uniform boundedness principle, the open mapping theorem and the closed graph theorem). This remark was made by I. Sadeqi and F.S. Kia [12] for FNLSs of type $(X, N, \wedge)$. Based on our results, these principles remain true without assuming $(N7)$ as in [12].

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A Unified Anti-Windup Technique for Fuzzy and Sliding Mode Controllers

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Abstract: This paper proposes the unified treatment of an anti-windup technique for fuzzy and sliding mode controllers. A back-calculation and tracking anti-windup scheme is proposed in order to prevent the zero error integrator wind-up in the structures of state feedback fuzzy controllers and sliding mode controllers. The state feedback sliding mode controllers are based on the state feedback-based computation of the switching variable. An example that copes with the position control of an electro-hydraulic servo-system is presented. The conclusions are pointed out on the basis of digital simulation results for the state feedback fuzzy controller.
Keywords: Anti-windup technique, electro-hydraulic servo-system, fuzzy control, saturation, sliding mode control, digital simulation.

1 Introduction

There are many situations in industrial control applications when a mismatch between the control signal (controller output) and the input of the process occurs. The saturation of controller output due to the functionality of the controller is such a natural process. The anti-windup techniques correct the controller output in the case of controllers with integral (I) component, but other components of the controllers can cause the saturation. Some discussions on the strong or weak impacts of the integrator wind-up and of saturation are presented in [1–5].

The anti-windup is applied to nonlinear systems including sliding mode control systems and fuzzy control systems, with representative examples given in [6–9]. A static anti-windup compensator for linear sliding mode controllers is designed in [10] on the basis of Linear Matrix Inequality (LMI) conditions derived from Lyapunov stability and L2-gain performance. The switching function of sliding mode controllers is modified in [11] to reduce the discontinuous component of the control signal during saturation. An adaptive anti-windup PID sliding mode scheme is proposed in [12]. The sliding surface for robust saturated sliding mode control is designed in [13] as a problem of root clustering. A dead-zone technique is employed in [9] in the framework of adaptive sliding mode control combined with fuzzy logic. The necessity of anti-windup measures in fuzzy control is pointed out in [14] and focused on Mamdani proportional-integral-fuzzy controllers. The fuzzy models of nonlinear models are involved in [15] in the design of piecewise fuzzy anti-windup dynamic output feedback controllers based on piecewise quadratic Lyapunov functions. Starting with a stabilizing dynamic output feedback fuzzy controller, an anti-windup block is designed in [16] to maximize the size of estimate of the domain of attraction, and sufficient Lyapunov-Krasovskii stability conditions are derived.

This paper suggests the application of a back-calculation and tracking anti-windup scheme to the zero error integrator in the framework of state feedback fuzzy controllers and state feedback sliding mode controllers. The specific feature of state feedback sliding mode controllers is the state feedback-based computation of the switching variable. The paper is supported by our previous results in fuzzy control [17–24], and proposes the unified treatment of anti-windup techniques in fuzzy and sliding mode controllers. This leads to good effects on the overall control system performance. The expression of the parameters of the anti-windup block is given, but these parameters can be optimized in terms of, for example, the optimal tuning of the anti-windup tracking gain carried out in [25].

This paper represents a step forward in the systematic design of fuzzy control systems, pointed out by Prof. Zadeh in [26] and [27]. The mathematics of fuzzy sets must be incorporated in the structure of fuzzy controllers by appropriate operators and parameters [28–30, 32, 32–34]. The model-based design using fuzzy models is emphasized in [35–45], but neural networks are used in [46–48]. This paper also represents a step forward in the systematic design of sliding mode control systems in the context of other popular techniques [49–53]. The parameters of both fuzzy and sliding mode controllers can be optimally tuned by means of appropriate optimization problems and algorithms [54–58].

The paper is organized as follows: Section 2 is dedicated to the modeling of state feedback sliding mode and fuzzy control systems. The proposed back-calculation and tracking anti-windup scheme is presented in Section 3. Section 4 applies the scheme to the position control of an electro-hydraulic servo-system and simulation results are included. The conclusions are outlined in Section 5.

2 Models of State Feedback Sliding Mode and Fuzzy Control Systems

The unified structure of state feedback sliding mode and fuzzy control systems is presented in Figure 1, where: w - the reference input, v - the disturbance input, u - the control signal, y - the controlled output, $e = w = y$ - the control error, $\mathbf{x}_p$ - the state vector of the process P, supposed to be observable and controllable, $\mathbf{x}_p \in \mathbf{R}^n$, ZEI - the zero error integrator (to obtain the zero steady-state value of the control error), x_R - the nominal integrator output, x_{RL}- the saturated (limited) integrator output, RB - the reference block, SB - the switching block, g - the switching variable, $\{k_R, k_w, T_i, u_0, \mathbf{k}_p^T\}$ - the parameters of the sliding mode controller (SMC), T stands for matrix transposition, $\mathbf{k}_p^T \in \mathbf{R}^{1\times n}$, $\{-x_L, x_L\}$ - the limits of the saturation element that belongs to ZEI, $x_L = \text{const} > 0$. The structure given in Figure 1 is obtained by the appropriate transformation and extension of the sliding mode control scheme presented in [59], the abbreviation FC indicates the fuzzy controller, and the detailed structure of SMC is illustrated in the lower part of Figure 1. Both controllers are presented as nonlinear blocks in Figure 1.

The P in the SMC is described by the n^{th} order state-space model:

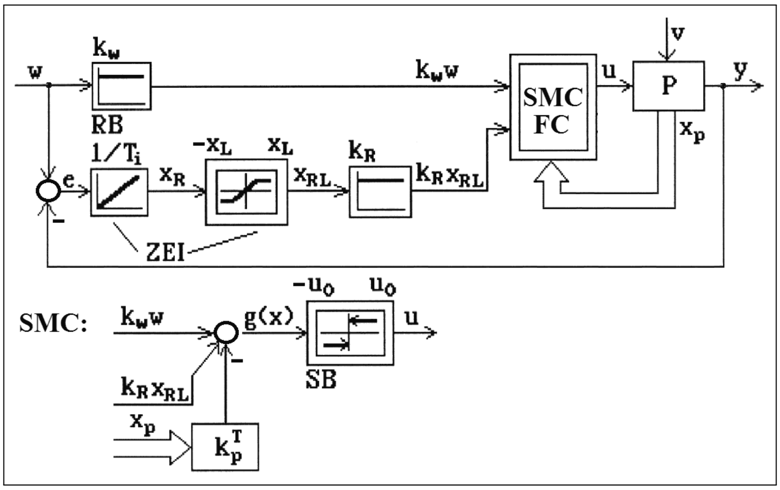

Figure 1: Unified structure of state feedback sliding mode and fuzzy control systems.

$$\begin{aligned} \dot{\mathbf{x}}_p &= \mathbf{A}_p\mathbf{x}_p + \mathbf{b}_p u + \mathbf{b}_{pv} v, \\ y &= \mathbf{c}_p^T \mathbf{x}_p, \end{aligned} \tag{1}$$

where $\mathbf{A}_p \in \mathbf{R}^{n\times n}$, $\mathbf{b}_p \in \mathbf{R}^{n\times 1}$, $\mathbf{b}_{pv} \in \mathbf{R}^{n\times 1}$, $\mathbf{c}_p^T \in \mathbf{R}^{1\times n}$. Introducing the extended state vector:

$$\mathbf{x} = [\ \mathbf{x}_p^T \quad x_R\ ]^T, \tag{2}$$

the state-space model of CP and ZEI is:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}u + \mathbf{b}_v v + \mathbf{b}_w w, \\ y &= \mathbf{c}^T\mathbf{x}, \end{aligned} \tag{3}$$

with the matrices:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_p & \mathbf{0} \\ -(1/T_i)\mathbf{c}_p^T & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} \mathbf{b}_p \\ 0 \end{bmatrix}, \mathbf{b}_v = \begin{bmatrix} \mathbf{b}_{pv} \\ 0 \end{bmatrix}, \mathbf{b}_w = \begin{bmatrix} \mathbf{0} \\ 1/T_i \end{bmatrix}, \mathbf{c}^T = [\ \mathbf{c}_p^T \quad 0\ ]. \tag{4}$$

The variable structure control law specific to SMC is of relay type:

$$u = u_0 \mathrm{sgn}(g(\mathbf{x})), g(\mathbf{x}) = -\mathbf{k}^T\mathbf{x} + k_w w, \mathbf{k}^T = [\ \mathbf{k}_p^T \quad -k_R\ ], \tag{5}$$

where the matrix $\mathbf{k}^T$ characterizes the switching hyper-plane and $u_0 = \mathrm{const} > 0$ is the absolute value of the control signal. The analysis of the control system in sliding mode is supported by the equivalent control method [60] resulting in the state-space equations of the control system in sliding mode:

$$\dot{\mathbf{x}} = \mathbf{A}^s\mathbf{x} + \mathbf{b}_v^s v + \mathbf{b}_w^s w + \mathbf{b}_{\dot{w}}^s \dot{w}, \tag{6}$$

with the matrices:

$$\begin{aligned} &\mathbf{A}^s = \begin{bmatrix} \mathbf{M}_p\mathbf{A}_p - [k_R/(T_i\mathbf{k}_p^T\mathbf{b}_p)]\mathbf{b}_p\mathbf{c}_p^T & \mathbf{0} \\ -(1/T_i)\mathbf{c}_p^T & 0 \end{bmatrix}, \mathbf{b}_v^s = \begin{bmatrix} \mathbf{M}_p\mathbf{b}_{pv} \\ 0 \end{bmatrix}, \mathbf{M}_p = \mathbf{I} - [1/(\mathbf{k}_p^T\mathbf{b}_p)]\mathbf{b}_p\mathbf{k}_p^T, \\ &\mathbf{b}_w^s = \begin{bmatrix} [k_R/(T_i\mathbf{k}_p^T\mathbf{b}_p)]\mathbf{b}_p \\ 1/T_i \end{bmatrix}, \mathbf{b}_{\dot{w}}^s = \begin{bmatrix} k_w/(\mathbf{k}_p^T\mathbf{b}_p) \\ 0 \end{bmatrix}, \end{aligned} \tag{7}$$

and $\mathbf{I}$ is the n^{th} order identity matrix. Equation (6) does not highlight the reaching mode, but its effect can be highlighted by the proper modification of the initial conditions. The sliding mode existence condition involves the equivalent control signal u_{eq}:

$$|u_{eq}| < u_0, u_{eq} = [1/(\mathbf{k}_p^T \mathbf{b}_p)] \cdot [k_R x_R - \mathbf{k}_p^T (\mathbf{A}_p \mathbf{x}_p + \mathbf{b}_{pv} v) + k_w \dot{w}]. \tag{8}$$

All above equations specific to SMC correspond to the case of ZEI that operates in its linear operating mode, i.e. the ZEI is not in saturation:

$$x_{RL} = x_R. \tag{9}$$

It is very convenient to use equation (6) in the analysis and design of the SMC as it is linear and it characterizes with an acceptable accuracy the behavior of a nonlinear control system (the sliding mode one). Similar models for the FC will be presented as follows.

The rule base of the continuous-time dynamic Takagi-Sugeno (T-S) fuzzy model of P consists of n_R rules, R^i, $i = 1...n_R$. Each rule is assigned to the following continuous-time state-space model in its consequent, namely to a local linear model of P:

$$\begin{aligned} \dot{\mathbf{x}}_p &= \mathbf{A}_{pi} \mathbf{x}_p + \mathbf{b}_{pi} u, \\ y &= \mathbf{c}_{pi}^T \mathbf{x}_p, i = 1...n_R, \end{aligned} \tag{10}$$

where $\mathbf{A}_{pi} \in \mathbf{R}^{n \times n}$, $\mathbf{b}_{pi} \in \mathbf{R}^{n \times 1}$, $\mathbf{c}_{pi}^T \in \mathbf{R}^{1 \times n}$, and the disturbance input is omitted for simplicity. The complete rule base of the continuous-time dynamic T-S fuzzy model of P is:

$$\begin{aligned} & R^i : \text{IF } z_1 \text{IS } LT_{z_1}^i \text{ AND } z_2 \text{ IS } LT_{z_2}^i \text{ AND ... AND } z_m \text{ IS } LT_{z_m}^i \\ & \text{THEN} \begin{cases} \dot{\mathbf{x}}_p = \mathbf{A}_{pi} \mathbf{x}_p + \mathbf{b}_{pi} u \\ y = \mathbf{c}_{pi}^T \mathbf{x}_p \end{cases}, i = 1...n_R, \end{aligned} \tag{11}$$

where $\mathbf{z} = [\ z_1 \quad z_2 \quad ... \quad z_m\ ]^T$ is the scheduling vector, i.e., the input vector, which contains the measurable variables of P, z_k, $k = 1...m$, and $LT_{z_k}^i$ are the input linguistic terms with the input membership functions $\mu_{z_k}^i(z_k)$.

Using the SUM and PROD operators in the inference engine and the weighted average defuzzification method, the firing strengths (of the rules) are:

$$w_i(\mathbf{z}) = \prod_{k=1}^{m} \mu_{z_k}^i(z_k), i = 1...n_R, \tag{12}$$

the normalized firing strengths are:

$$h_i(\mathbf{z}) = w_i(\mathbf{z}) / [\sum_{i=1}^{n_R} w_i(\mathbf{z})], i = 1...n_R, \tag{13}$$

and the continuous T-S fuzzy model of P is expressed in the state-space form:

$$\begin{aligned} \dot{\mathbf{x}}_p &= \sum_{i=1}^{n_R} [h_i(\mathbf{z})(\mathbf{A}_{pi} \mathbf{x}_p + \mathbf{b}_{pi} u)], \\ y &= \sum_{i=1}^{n_R} [h_i(\mathbf{z}) \mathbf{c}_{pi}^T \mathbf{x}_p]. \end{aligned} \tag{14}$$

In parallel distributed compensation (PDC) the structure of the FC model matches the structure of the fuzzy model of P given in (11). Considering the absence of the blocks RB and ZEI, the PDC controller for the system (11) is:

$$u = -\sum_{i=1}^{n_R} [h_i(\mathbf{z})\mathbf{f}_i^T \mathbf{x}_p], \tag{15}$$

and the goal of FC design is to obtain the gain matrices $\mathbf{f}_i^T, i = 1...n_R, \mathbf{f}_i^T \in \mathbf{R}^{1\times n}$, of the nonlinear state-feedback control law (15) such that the closed-loop system is stable and eventually robust. As outlined in [61], many design problems derive the least conservative conditions related to the condition:

$$\sum_{i=1}^{n_R}\sum_{j=1}^{n_R} [h_i(\mathbf{z})h_j(\mathbf{z})\mathbf{\Gamma}_{ij}] < 0, \mathbf{\Gamma}_{ij} = \mathbf{\Gamma}_{ij}{}^T. \tag{16}$$

Considering the fuzzy control system structure according to Figure 1, the control law of the FC given in (15) is modified as follows in the specific case of ZEI that operates in its linear operating mode:

$$u = k_w w + k_R x_R - \sum_{i=1}^{n_R} [h_i(\mathbf{z})\mathbf{f}_i^T \mathbf{x}_p]. \tag{17}$$

Using (2), (17) and Figure 1 in (14), the state-space equations of the fuzzy control system are:

$$\dot{\mathbf{x}} = \mathbf{A}^s \mathbf{x} + \mathbf{b}_w^s w, \tag{18}$$

with the matrices:

$$\begin{aligned}
\mathbf{A}^s &= \begin{bmatrix} \sum\limits_{i=1}^{n_R} [h_i(\mathbf{z})\mathbf{A}_{pi}] - \left[\sum\limits_{i=1}^{n_R} [h_i(\mathbf{z})\mathbf{b}_{pi}]\right] \cdot \left[\sum\limits_{i=1}^{n_R} [h_i(\mathbf{z})\mathbf{f}_i^T]\right] & k_R \sum\limits_{i=1}^{n_R} [h_i(\mathbf{z})\mathbf{b}_{pi}] \\ -(1/T_i) \sum\limits_{i=1}^{n_R} [h_i(\mathbf{z})\mathbf{c}_{pi}^T] & 0 \end{bmatrix}, \\
\mathbf{b}_w^s &= \begin{bmatrix} k_w \sum\limits_{i=1}^{n_R} [h_i(\mathbf{z})\mathbf{b}_{pi}] \\ 1/T_i \end{bmatrix}.
\end{aligned} \tag{19}$$

The models (6) (of the sliding mode control system) and (18) (of the fuzzy control system) are similar. Moreover, the gain matrices $\mathbf{f}_i^T$ of FC are similar to the gain matrix $\mathbf{k}_p^T$ of SMC (illustrated in Figure 1). This justifies the unified treatment of anti-windup techniques for fuzzy and sliding mode controllers. However, sliding mode and fuzzy control systems have different structures; although the fuzzy control system is more complicated, it is not constrained to enter the sliding mode related to the sliding mode existence condition.

If ZEI enters saturation, i.e.:

$$x_{RL} = x_{R\,\mathrm{lim}}, x_{R\,\mathrm{lim}} \in \{-x_L, x_L\}, \tag{20}$$

the updated expression of the switching variable is:

$$g(\mathbf{x}_p) = -\mathbf{k}_p^T \mathbf{x}_p + k_w w + k_R x_{R\,\mathrm{lim}}, \tag{21}$$

and the equivalent control method leads to the updated equivalent control signal:

$$u_{eq} = [1/(\mathbf{k}_p^T \mathbf{b}_p)] \cdot [-\mathbf{k}_p^T (\mathbf{A}_p \mathbf{x}_p + \mathbf{b}_{pv} v) + k_w \dot{w}], \tag{22}$$

and to the updated state-space equations of P in sliding mode:

$$\dot{\mathbf{x}}_p = \mathbf{A}_p^s \mathbf{x}_p + \mathbf{b}_{pv}^s v + \mathbf{b}_{p\dot{w}}^s \dot{w}, \tag{23}$$

with the matrices:

$$\mathbf{A}_p^s = \mathbf{M}_p\mathbf{A}_p, \mathbf{b}_{pv}^s = \mathbf{M}_p\mathbf{b}_{pv}, \mathbf{b}_{p\dot{w}}^s = [k_w/(\mathbf{k}_p^T\mathbf{b}_{pv})]\mathbf{b}_p. \tag{24}$$

Hence, at least one negative effect on control system behavior can be observed when the saturation of the ZEI occurs or when the integrator output is maintained in saturation/outside the limits on a too large time interval. This consists in the difficult fulfillment of the sliding mode existence condition due to the different expressions of u_{eq} in (8) and (22). Therefore, anti-windup techniques are necessary not just to get ZEI out of saturation, but just to avoid exaggerate exceeds of limitation.

3 Back-Calculation and Tracking Anti-Windup Scheme

The state-space equation of I that belongs to ZEI is:

$$\dot{x}_R = (1/T_i)e. \tag{25}$$

Equation (25) is also kept in the presence of windup, but with the substitution of e with another **I** input, e_L, chosen such that to keep ZEI in saturation, i.e., (9) is applied, therefore:

$$\dot{x}_{RL} = (1/T_i)e_L. \tag{26}$$

Equation (25) holds in the absence of windup, and equation (26) corresponds to the presence of windup. Subtracting (26) from (25) results in:

$$e_L = e - T_i(\dot{x}_R - \dot{x}_{RL}). \tag{27}$$

The structure of ZEI with back-calculation and tracking anti-windup scheme is built using (27) and given in Figure 2. BCT in Figure 2 represents the back-calculation and tracking block, with pure derivative character, modeled by the transfer function:

$$H_{BCT}(s) = T_i s. \tag{28}$$

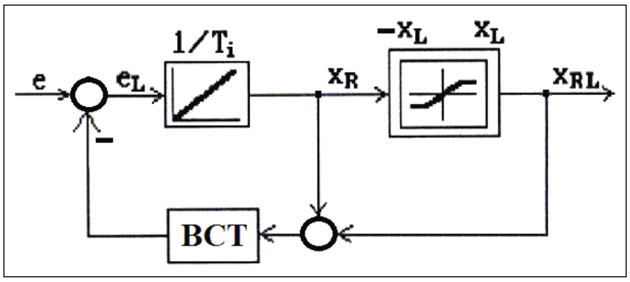

Figure 2: Unified structure of ZEI with back-calculation and tracking anti-windup scheme.

The proposed anti-windup technique has a shortcoming, namely it does not operate if x_R enters saturation, and it is constant, or if the difference $(x_R - x_{RL})$ is constant. Therefore, the modified BCT block (MBCT) is of lead-lag type with the following transfer function:

$$H_{MBCT}(s) = (T_i s + k_{AW})/(1 + T_{AW}s), \tag{29}$$

where the time constant T_{AW} (a small value) is necessary in order to make possible the implementation of MBCT. The anti-windup tracking gain k_{AW} can take any positive value. However, an as small as possible value of k_{AW} is recommended because: (i) the strong feedback derivative action already determines the integrator to stay very close to the saturation limit, (ii) a large value of k_{AW} could lead to stability problems investigated in [62].

4 Position Control Application. Simulation Results

An example of nonlinear electro-hydraulic system [63] is taken into consideration in order to illustrate the advantages of the proposed back-calculation and tracking anti-windup scheme in the context of state feedback sliding mode control. The simplified structure of the electro-hydraulic system meant for position (y) control is presented in Figure 3. The parameters of P are: $g_0 = 0.0625$, $T_{i1} = 0.002$s, $T_{i2} = 0.065$s, $x_L = 0.5$.

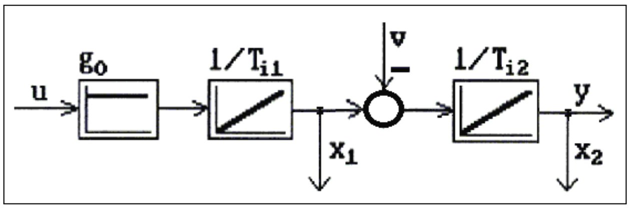

Figure 3: Simplified structure of P.

The design of the SMC is performed according to [59], but for $n = 2$, and the following values of controller parameters are obtained: $k_{x1} = 1$, $k_{x2} = 0.13$, $\mathbf{k}_p^T = [\ k_{x1} \quad k_{x2}\ ] = [\ 1 \quad 0.13\ ]$, $k_w = 0.13$, $k_R = 32.5$, $T_i = 0.1$s, $u_0 = 15$. The overall control system is referred to as electro-hydraulic servo-system.

The designed sliding mode control system was tested with respect to the modifications of w using the nonlinear model of P. Figure 4 gives the control system response (the control system output y with continuous line, w with dash dotted line, x_{RL} with continuous line) without limitations imposed to ZEI. Figure 4 shows that y tracks the imposed w. Figure 5 gives the

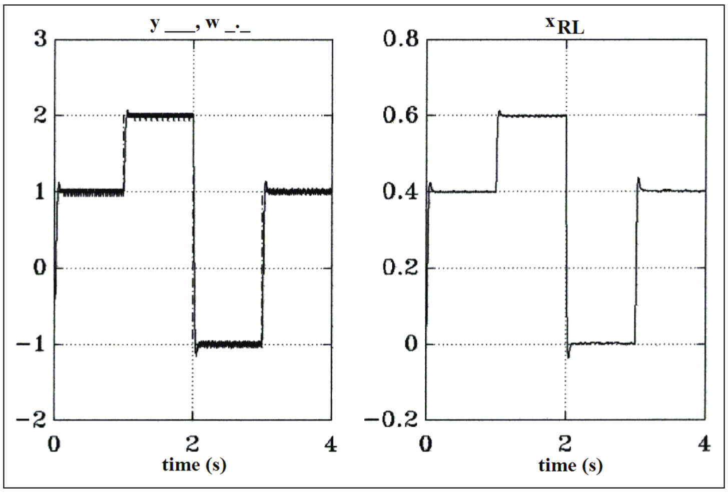

Figure 4: Control system response without limitations.

control system response without anti-windup technique. The harmful effect of the limitation is illustrated by the fact that y does not track anymore w, and x_R stays during a relatively long time period in limitation within the time interval $[1\text{s}, 3\text{s}]$.

Figure 6 gives the control system response with classical back-calculation and tracking anti-windup scheme using the same w and conditions as in Figure 4. The control system performance is improved in comparison with the case without back-calculation and tracking anti-windup scheme.

Figure 7 gives the control system response that incorporates the proposed modified back-calculation and tracking anti-windup scheme using the same w and conditions as in Figure 4, and

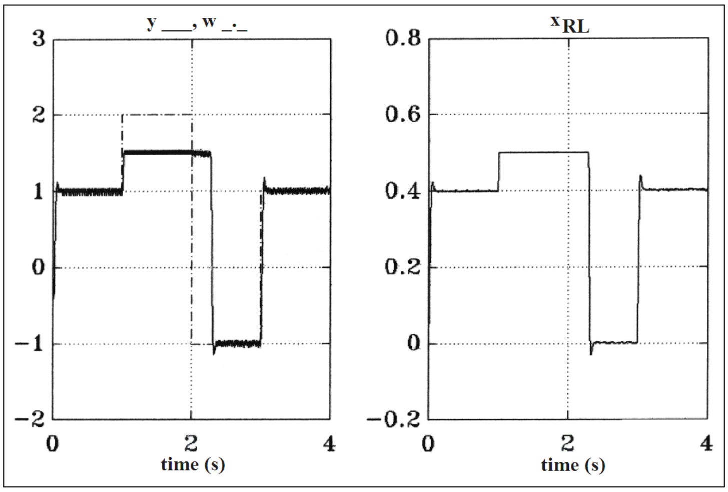

Figure 5: Control system response without BCT.

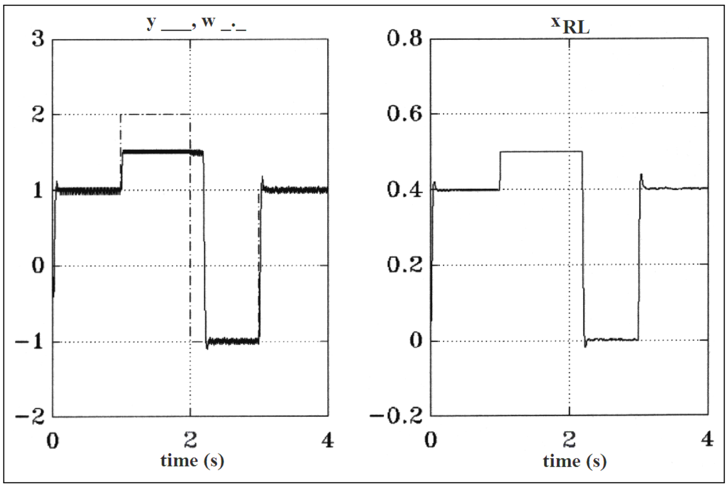

Figure 6: Control system response with BCT.

for $k_{AW} = 0.02$ and $T_{AW} = 0.01$s. Figure 7 shows the control system performance improvement compared to all previous cases. One relatively minor shortcoming concerns the slightly larger effect of the chattering phenomenon.

5 Conclusions

The paper has proposed an approach to the unified treatment of anti-windup in fuzzy and sliding mode controllers. The unified models of fuzzy control systems and sliding mode control systems have been suggested.

An analysis of the possibility to apply the back-calculation and tracking anti-windup scheme to the zero error integrator belonging to a state feedback sliding mode controller has been carried out. A modified back-calculation and tracking anti-windup scheme applicable to these controller structures has been suggested.

The future work will be dedicated to the discrete time formulation of the controllers and of the anti-windup schemes and to the stability analysis because the control systems stability can be affected or not by windup. The parameters of the anti-windup schemes will be tuned by

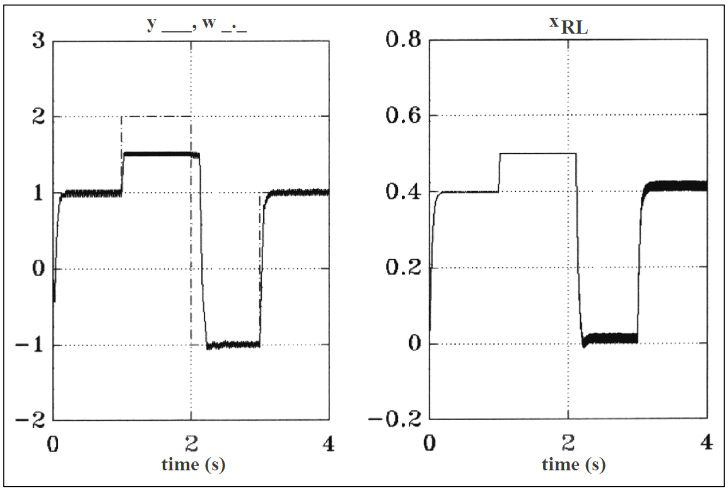

Figure 7: Control system response with MBCT.

several optimization algorithms in the context of appropriate optimization problems.

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An Improved Attribute Reduction Algorithm based on Granular Computing

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Abstract: Granular computing is a new intelligent computing method based on problem solving, information processing and pattern classification. Granular computing based attribute reduction method is an important application of Granular computing. These algorithms are mostly based on reduction core. However, some information systems may have no reduction core, especially in the actual application data. For this case, those algorithms are powerless. In this paper, an improved reduction algorithm based on granular computing is proposed. The algorithm is validated by the experimental result.
Keywords: attribute reduction, granular computing, rough set, attribute significance.

1 Introduction

Granular computing is a method for analysis of multi-layer granular structure based on problem solving, pattern classification and information processing. It's also a newly cross discipline among rough set theory, fuzzy set theory, data mining and artificial intelligence. With less than 20 years' development, granular computing has already made remarkable achievements and great contribution to the field of computer science [1, 2]. Through rapid development of society and continuous progress in science and technology, a variety of data is increasing gradually, and then we entered the so-called "Big Data Time". The main goal of data mining is to find potential, desired and useful knowledge from those big data. Rough set theory is an efficient mathematical tool to deal with imprecise, incomplete and inconsistent data. It has already made great strides in its theory and has been widely used in practical application.

Attribute reduction is the main content of rough set theory. The core task of attribute reduction is that dimensionality and storage space may be reduced under the condition of maintaining classification capacity, so as to improve the efficiency of system classification [3, 4]. Therefore, it is not only the hot spot of intelligence computing, but also the important task of information processing.

In 1979, professor L.A. Zadeh discussed the theory of fuzzy information granulation in his paper "Fuzzy Sets and Information Granularity", and first proposed the concept of information granulation. Then, professor J.R. Hobss of Stanford University introduced granularity theory in his paper "Granularity" published on International Joint Conference on Artificial Intelligence held in Los Angeles [5]. The granularity theory is presented firstly. The idea of granularity theory is that the bigger, whole, unresolved questions can be broken into several smaller ones by granulating, and these small questions can be combined into the bigger, whole questions. In 1990, the Chinese scholar Zhang Bo and Zhang Ling proposed the theory of quotient space based on problem solving [6]. They thought that human beings can analyze the same problem from different granulation, and make an easy conversion in different knowledge granularity. If people can formalize the analysis process to make the computer possess the ability, it will greatly improve the development of artificial intelligence. Furthermore, professor L.A. Zadeh raised the theory of Computing with Words in his paper "Fuzzy Logic=Computing with Words", and thus the fuzzy granularity theory was born [7]. This theory is to do fuzzy reasoning and judgments by using natural language, so as to realize the fuzzy intelligent control. In the same year, when Professor L.Y. Lin visited in Professor Zadeh's Key Laboratory of UC-Berkeley University, he presented the subject "Granular Computing" and got approval from Zadeh, it marked the birth of granular computing.

Professor Miao, et al. [8] gave the definition of knowledge granularity and knowledge discernibility in fuzzy set theory model, and pointed out the relationship between knowledge granularity and knowledge discernibility: the smaller knowledge granularity is, the stronger knowledge distinguishable ability is; on the contrary, the bigger knowledge granularity is, the weaker knowledge distinguishable ability is. Reference [9] defined the concept of the difference of granularity and granularity entropy on the basis of fuzzy set's algebraic method and information theory approach, and proposes attribute reduction algorithm based on granular computing. Reference [10] presented attribute reduction algorithm based on Granular Computing, using the equivalent relation in rough set to construct granule, and attribute significance is regarded as heuristic information. Reference [11] put forward attribute reduction method based on model of granular computing in information systems. Reference [12] proposed an improvement of attribute reduction algorithm based on Granular Computing. This algorithm is to get attribute core using discernibility matrix, and then make attribute reduction based on attribute significance as heuristic information. Reference [13] proposed an incomplete order decision table reduction algorithm based on granular computing.

These reduction methods based on granular computing are mainly first to calculate reduction core of system, then get reduction based on core. However, in practical application, some information systems may have no reduction core. In this case, this paper proposes an improved reduction algorithm based on attribute significance of granular computing, and numerical experiments show the effectiveness of the algorithm.

2 Basic Concepts of Rough Sets

2.1 Rough sets

Let a quadruple $S = (U, A, V, f)$ be an information systems (IS), in which $U = \{x_1, x_2, \cdots, x_n\}$ is a non-empty finite set called the domain of discourse; $A = \{a_1, a_2, \cdots, a_m\}$ is a non-empty and finite set of attributes; V is a set of attribute values domain, $V = \underset{a \in A}{\cup} V_a$; $f : U \times A \to V$ is a mapping, each attribute of the object in the domain of discourse by the mapping has a corresponding information value, i.e. $\forall a \in A, x \in U, f(x, a) \in V_a$. If the attributes set A is composed of condition attributes set C and decision attributes set D, the quadruple $S = (U, A, V, f)$ is also called decision information system (DIS). The information system, also known as knowledge representation system, is the main expression of knowledge of rough sets. It is simply expressed in (U, A). If P is a subset of attributes set A , each subset $P \subseteq A$ determines a binary indistinguishable relation $IND(P)$, $IND(P) = \{(u,v) \in U \times U | \forall a \in P, a(u) = a(v)\}$. A set $X \subseteq U$ represent a concept and partition included by $IND(P)$ is called a knowledge base and denoted by $U/IND(P)$. In particular, the partition $U/IND(P) = \{Y_1, Y_2, \cdots, Y_k\}$ is the knowledge base of decision classes.

A knowledge base (U,R) is also called an approximation space, where U is the domain of discourse and R is an equivalence relation on U. Let $X \subseteq U$and$R \subseteq A$, the sets $\underline{R}(X) = \{x \in U | [x]_R \subseteq X\}$ and $\bar{R}(X) = \{x \in U | [x]_R \cap X \neq \phi\}$ are respectively called lower approximation set and upper approximation set. Where $[x]_R$ refers to an equivalence class of $IND(P) = \cap IND(R)$ determined by element x. If $\underline{R}(x) = \bar{R}(x)$, then X is called a definable set on U; If $\underline{R}(x) \neq \bar{R}(x)$, then X is called a rough set on U.

If $IND(R) \neq IND(R - |a|)$, then a is indispensable in the set R, otherwise a is dispensable. If every $a \in R$ is indispensable, then R is called independent. Let $Q \subseteq P$, that is, Q is a subset of P, if Q is independent and $IND(Q) = IND(P)$, then Q is a reduction of P, denoted as $Q = red(P)$. The union set of indispensable attribute in the set A is called a core set, denoted as $core(P)$, $core(P) = \cap red(P)$.

2.2 Knowledge granulation and partition

Definition 1[8]. Let (U,R) be an approximation space, $P \in R$ is an equivalence relation on U, called knowledge. The approximation space is also called knowledge base. The equivalence class $[x]_P = \{x \in U, (x_i, x_j) \in P\}$ is called knowledge granule. The quotient set $U/P = \{[x\,]_P | x \in U\}$ is called a $P - granularity$ partition. The granularity of knowledge is defined as $GD(P)$,

$$GD(P) = \frac{|P|}{|U \times U|} = \frac{|P|}{|U|^2} \tag{1}$$

Where $|P|$ denotes the cardinality of the set $P \subseteq U \times U$.

The granularity of knowledge P can express its distinguishable ability. For $\forall u, v \in U$, if $(u,v) \in P$, then they belong to the same equivalence class, i.e. they are indistinguishable. The knowledge P's discernibility could be defined as $Dis(P)$, $Dis(P) = 1 - GD(P)$. In general, the greater the granularity is, the weaker the distinguishable ability will be, vice versa.

Theorem 1[8]. Let $P \in R$ be a knowledge of knowledge base $K = (U,R)$, if $U/P = \{X_1, X_2, \cdots, X_n\}$, then

$$GD(P) = \left(\sum_{i=1}^{n} |X_i|^2 \right) \bigg/ |U|^2. \tag{2}$$

Property 1. Let $P, Q \in R$ be an equivalence relations on U, $U/P = \{X_1, X_2, \cdots, X_n\}$, $U/Q = \{Y_1, Y_2, \cdots, Y_n\}$, if $P = Q$, then $GD(P) = GD(Q)$ and $Dis(P) = Dis(Q)$; if $P \prec Q$, $GD(P) < GD(Q)$ and $Dis(Q) < Dis(P)$.

proof. (a) If $P = Q$, then $m = n$, $X_i = Y_i$, so $GD(P) = GD(Q)$, $Dis(Q) = Dis(P)$.

(b) If $P \prec Q$, then $|P| < |Q|$, so $GD(P) < GD(Q)$. Since $Dis(P) = 1 - GD(P)$, we could observe that $Dis(Q) < Dis(P)$.

Property 2. Let $P \in R$ be an equivalence relation on U, $U/P = \{X_1, X_2, \cdots, X_n\}$, if the equivalence relation P divides from knowledge granules in U/R, then $GD(P) \leq GD(R)$, $Dis(R) \leq Dis(P)$.

Proof. We suppose that the knowledge granule X_i from U/R is divided into two knowledge granules X_{i1} and X_{i2}, that is $X_i = X_{i1} \cup X_{i2}$ and $X_{i1} \cap X_{i2} = \emptyset$, $U/P = \{X_1, X_2, \cdots, X_{i-1}, X_{i1}, X_{i2}, X_{i+1}, \cdots, X_n\}$, so

$$GD(R) = \left(\sum_{j=1}^{n} |X_j|^2 \right) \bigg/ |U|^2$$

$$= \left(\sum_{j=1}^{i-1} |X_j|^2\right) \Big/ |U|^2 + |X_i|^2 \Big/ |U|^2 + \left(\sum_{j=i+1}^{n} |X_j|^2\right) \Big/ |U|^2$$
$$= \left(\sum_{j=1}^{i-1} |X_j|^2\right) \Big/ |U|^2 + [|X_{i1}| + |X_{i2}|]^2 \Big/ |U|^2 + \left(\sum_{j=i+1}^{n} |X_j|^2\right) \Big/ |U|^2$$
$$\geq \left(\sum_{j=1}^{i-1} |X_j|^2\right) \Big/ |U|^2 + \left[|X_{i1}|^2 + |X_{i2}|^2\right] \Big/ |U|^2 + \left(\sum_{j=i+1}^{n} |X_j|^2\right) \Big/ |U|^2$$
$$= GD(P),$$

$Dis(R) = 1 - GD(R) \leq 1 - GD(P) = Dis(P)$.

Property 3. Let (U, R) be a knowledge base and $P \in R$ be an equivalence relation on U, $U/R = \{X_1, X_2, \cdots, X_n\}$, Q is the union of knowledge granules in U/R, then $GD(R) \leq GD(Q)$, $Dis(Q) \leq Dis(R)$.

Proof. We suppose that the knowledge granule X_k is the union of X_i and $X_i + 1$, then $U/Q = \{X_1, X_2, \cdots, X_{i-1}, X_k, X_{i+2}, \cdots, X_n\}$, so

$$GD(R) = \left(\sum_{j=1}^{n} |X_j|^2\right) \Big/ |U|^2$$
$$= \left(\sum_{j=1}^{i-1} |X_j|^2\right) \Big/ |U|^2 + |X_i|^2 \Big/ |U|^2 + |X_{i+1}|^2 \Big/ |U|^2 + \left(\sum_{j=i+2}^{n} |X_j|^2\right) \Big/ |U|^2$$
$$\leq \left(\sum_{j=1}^{i-1} |X_j|^2\right) \Big/ |U|^2 + [|X_i| + |X_{i+1}|]^2 \Big/ |U|^2 + \left(\sum_{j=i+2}^{n} |X_j|^2\right) \Big/ |U|^2$$
$$= \left(\sum_{j=1}^{i-1} |X_j|^2\right) \Big/ |U|^2 + |X_k|^2 \Big/ |U|^2 + \left(\sum_{j=i+2}^{n} |X_j|^2\right) \Big/ |U|^2$$
$$= GD(Q),$$

$Dis(Q) \leq Dis(R)$.

Property 4. Let $S = (U, A, V, f)$ be an information system,$P, Q \subseteq A$,

(1) If $P \Rightarrow Q$, then $GD(P) \leq GD(Q)$;

(2) If $P \Leftrightarrow Q$, then $GD(P)$=$GD(Q)$.

Proof. (1) If $P \Rightarrow Q$, then $IND(P) \subseteq IND(Q)$, this is $|IND(P)| \leq |IND(Q)|$. On the other hand, $GD(P) = GD(IND(P)) = |IND(P)| \Big/ |U|^2$, and $GD(Q) = GD(IND(Q)) = |IND(Q)| \Big/ |U|^2$, so $GD(P) \leq GD(Q)$.

(2) If $P \Leftrightarrow Q$, then $P \Rightarrow Q$ and $Q \Rightarrow P$. By (1), we could see that $GD(P) \leq GD(Q)$ and $GD(Q) \leq GD(P)$, so$GD(P)$=$GD(Q)$.

Property 5. Let $S = (U, A, V, f)$ be an information system,$P, Q \subseteq A$,

(1) If $P \Rightarrow Q$, then $Dis(P) \geq Dis(Q)$;

(2) If $P \Leftrightarrow Q$, then $Dis(P)$=$Dis(Q)$.

Proof. It follows immediately from Definition 1 and Property 4.

Deduction 1. Let $S = (U, A, V, f)$ be an information system, if $P \subseteq Q \subseteq A$, then $GD(Q) \leq GD(P)$ and $Dis(Q) \geq Dis(P)$.

Remark. Deduction 1 illustrates that for the subset of A, when the attribute number increased, the knowledge granularity is reduced, thus, the discernibility is increased.

3 Attribute reduction algorithm based on attribute significance

Definition 2[8]. Let $S = (U, A, V, f)$ be an information system, the attribute significance could be defined as $Sig_{A-\{a\}}(a)$,

$$Sig_{A-\{a\}}(a) = GD(A - \{a\}) - GD(A). \tag{3}$$

Remark. In an information system $S = (U, A, V, f)$, the attribute significance of each attribute $a \in A$ could be measured by knowledge granularity.

Definition 3[8]. Let $S = (U, A, V, f)$ be an information system, C is a subset of A, $C \subseteq A$, for $\forall a \in A - C$, the attribute significance of attribute a relative to attribute set C could be defined as $Sig_C(a)$,

$$Sig_C(a) = GD(C) - GD(C \cup \{a\}). \tag{4}$$

Remark. Definition 3 illustrates that the attribute significance of attribute a relative to attribute set C could be measured by change of the knowledge granularity. When a attribute is added to attribute set C, C's knowledge granularity may change. If C's knowledge granularity change, then attribute a is indispensable.

Definition 4[8]. Let $S = (U, A, V, f)$ be an information system, $a \in A$, if $GD(A - \{a\}) = GD(A)$, then attribute a is dispensable, otherwise, attribute a is indispensable. If every $a \in A$ is indispensable, then A is called independent.

Definition 5[8]. Let $S = (U, A, V, f)$ be an information system, $P \subseteq A$, if P is independent and $GD(P) = GD(A)$, then P is a reduction of A, denoted as $red(A)$. The union set of indispensable attribute in the set A is called a core set, denoted as $core(P)$, $core(P) = \cap red(P)$.

Property 6. Attribute a is indispensable, if and only if

$$Sig_{A-\{a\}}(a) > 0. \tag{5}$$

Proof. $\Rightarrow$ If attribute a is indispensable, then $GD(A - \{a\}) \neq GD(A)$. As we know that $GD(A - \{a\}) \geq GD(A)$, so $Sig_{A-\{a\}}(a) = GD(A - \{a\}) - GD(A) > 0$.

$\Leftarrow$ Obviously.

Property 7. $Core(A) = \cup \{a \in A | \, Sig_{A-\{a\}}(a) > 0\}$.

Proof: It follows immediately from Definition 3 and Property 6.

Remark. The attribute significance from the perspective of knowledge granularity provides a method of attribute reduction: We could judge the significance of attribute a by discussing whether $GD(A - \{a\})$ is equal to $GD(A)$. If $GD(A - \{a\}) = GD(A)$ŁŹthen a is dispensable, otherwise a is indispensable. Thus we could obtain the reduction core $Core(A)$. Next calculate the significance of the rest attribute relative to $Core(A)$. If $GD(Core(A) \cup a) = GD(A)$, then the set $Core(A) \cup a$ is the reduction of the information system, where $a = \{a \in A - Core(A\,) | \max Sig_{Core(A)}(a)\}$.

Algorithm 1:

Input: An information system $S = (U, A, V, f)$, where $U = \{x_1, x_2, \cdots, x_n\}$, $A = \{a_1, a_2, \cdots, a_m\}$.

Output: $red(A)$ and $Core(A)$ // the sets of reductions and core.

Step 1: For $i = 1$, $i \leq n$, $++i$; $j = 1$, $j \leq m$, $++j$ begin, calculate $GD(A)$// the knowledge granularity of attribute set A.

Step 2: Calculate $Sig_{A-\{a\}}(a)$ // the significance of attribute $a \in A$.

Step 3: Calculate $Core(A)$, $Core(A) = \{a \in A | \, Sig_{A-\{a\}}(a) > 0\}$.

Step 4: If $GD(Core(A)) = GD(A)$, output $red(A) = Core(A)$, end.

If $GD(Core(A)) > GD(A)$, turn next.

Step 5: Calculate $\max\limits_{b \in B} Sig_{Core(A)}(b)$, $B = A - Core(A)$ // the significance of attribute $b \in B = A - Core(A)$ for $Core(A)$.

Step 6: If $GD(Core(A) \cup a\prime) = GD(A)$, output $red(A) = Core(A) \cup b$ and $Core(A)$, end.

If $GD(Core(A) \cup b) > GD(A)$, repeat step 5,

Step 7: Calculate $\max\limits_{c \in C} Sig_{Core(A) \cup b}(b) \cup (Core(A) \cup b)$ // the significance of attribute $c \in C = A - Core(A) \cup b$ for $Core(A) \cup b$,

Step 8: For $j = 1$ to $|C|$, repeat step 7 until the knowledge granularity is equal to $GD(A)$

Step 9: Output $red(A) = Core(A) \cup b \cup c \cup \cdots$ and $Core(A)$.

Example 1. Let $S = (U, A, V, f)$ be an information system (Table 1). There are 6 objects and 4 attributes, where $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = \{a_1, a_2, a_3, a_4\}$. Calculate the reduction of the system.

Table1. An information system

U	a_1	a_2	a_3	a_4
x_1	1	0	2	2
x_2	0	1	1	1
x_3	2	0	0	1
x_4	1	1	0	2
x_5	2	2	0	0
x_6	2	1	1	1

It is easy to calculate that

$U/A = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $GD(A) = \left(\sum\limits_{i=1}^{n} |X_i|^2\right) \Big/ |U|^2 = \frac{6}{36} = \frac{1}{6}$,

$U/A - \{a_1\} = \{x_1, \{x_2, x_6\}, x_3, x_4, x_5\}$, $GD(A - \{a_1\}) = \frac{8}{36}$,

$Sig_{A-\{a_1\}}(a_1) = GD(A - \{a_1\}) - GD(A) = \frac{8}{36} - \frac{6}{36} = \frac{2}{36}$,

$U/A - \{a_2\} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $GD(A - \{a_2\}) = \frac{6}{36}$,

$Sig_{A-\{a_2\}}(a_2) = GD(A - \{a_2\}) - GD(A) = \frac{6}{36} - \frac{6}{36} = 0$,

$U/A - \{a_3\} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $GD(A - \{a_3\}) = \frac{6}{36}$,

$Sig_{A-\{a_3\}}(a_3) = GD(A - \{a_3\}) - GD(A) = \frac{6}{36} - \frac{6}{36} = 0$,

$U/A - \{a_4\} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $GD(A - \{a_4\}) = \frac{6}{36}$,

$Sig_{A-\{a_4\}}(a_4) = GD(A - \{a_4\}) - GD(A) = \frac{6}{36} - \frac{6}{36} = 0$.

Based on the result above, we can see that the core of the system is $Core(A) = \left\{a \in A |\, Sig_{A-\{a\}}(a) > 0\right\} = \{a_1\}$, $GD(Core(A)) = \frac{14}{36} > \frac{6}{36} = GD(A)$.

Let a set $B = A - Core(A) = \{a_2, a_3, a_4\}$, calculate the significance of the rest attribute relative to $Core(A)$:

$Sig_{Core(A)}(a_2) = GD(Core(A)) - GD(Core(A) \cup a_2) = \frac{14}{36} - \frac{6}{36} = \frac{8}{36}$,

$Sig_{Core(A)}(a_3) = GD(Core(A)) - GD(Core(A) \cup a_3) = \frac{14}{36} - \frac{8}{36} = \frac{6}{36}$,

$Sig_{Core(A)}(a_4) = GD(Core(A)) - GD(Core(A) \cup a_4) = \frac{14}{36} - \frac{10}{36} = \frac{4}{36}$.

Since $\max\limits_{b \in B} Sig_{Core(A)}(b) = \frac{8}{36}$, $U/Core(A) \cup a_2 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $GD(Core(A) \cup a_2) = GD(A)$, we can see that $Core(A) \cup a_2 = \{a_1, a_2\}$ is the reduction, that is $red(A) = \{a_1, a_2\}$, $Core(A) = \{a_1\}$.

4 An improved reduction algorithm

The reduction of an information system is not the only, some may have more than one reductions. But the reduction results may not be able to get reduction core, especially in the actual application data. For this case, Algorithm 1 is powerless. Now we will improve the algorithm, which can deal with the system with reduction core and no reduction core.

In an information system $S = (U, A, V, f)$, if $Sig_{A-\{a\}}(a) = 0$, then then a is dispensable. For $\forall a \in A$, if $Sig_{A-\{a\}}(a) = 0$, then the system has no reduction core.

Definition 6. Let $S = (U, A, V, f)$ be an information system, P is a subset of A, $P \subseteq A$, the attribute significance of P relative to A could be defined as $Sig_{A-P}(P)$,

$$Sig_{A-\{P\}}(P) = GD(A - P) - GD(A). \tag{6}$$

Particularly, if $P = A$, $Sig_{\emptyset}(P)$ is represented as $Sig(P)$, and $Sig(P) = Sig_{\emptyset}(P) = GD(\emptyset) - GD(A) = 1 - GD(P) = Dis(P)$. Where $GD(\emptyset) = 1$ ($since\ U/IND(\emptyset) = \{U\}$).

Algorithm 2:

Input: An information system $S = (U, A, V, f)$, where $U = \{x_1, x_2, \cdots, x_n\}$, $A = \{a_1, a_2, \cdots, a_m\}$.

Output: $red(A)$ and $Core(A)$ // the sets of reductions and core.

Step 1: For $i = 1$, $i \leq n$, $++i$; $j = 1$, $j \leq m$, $++j$ begin, calculate $GD(A)$.// the knowledge granularity of attribute set A

Step 2: Calculate $Sig_{A-\{a\}}(a)$. // the significance of attribute $a \in A$

If $Sig_{A-\{a\}}(a) \neq 0$ // the system has reduction core, turn step 8;

If $Sig_{A-\{a\}}(a) = 0$ // the system has no reduction core, next;

Step 3: Calculate $Sig_{A-\{a_i,a_j\}}(a_i, a_j)$, $1 \leq i \neq j \leq m$ // the significance of the combination of any two attributes in A,

Step 4: For $i = 1$, $i \leq n$, $++i$; $j = 1$, $j \leq m$, $++j$, find out

$red\prime(A) = \left\{ (a_i, a_j) | \max\limits_{1 \leq i \neq j \leq m} Sig_{A-\{a_i,a_j\}}(a_i, a_j) \right\}$. // the suboptimal reduction of the system

Step 5: If $GD(red\prime(A)) = GD(A)$, then output the reduction $red(A) = red\prime(A)$, end. If $GD(red\prime(A)) > GD(A)$, turn next;

Step 6: Calculate $\max\limits_{b \in B} Sig_{red\prime(A)}(b)$, $B = A - red\prime(A)$. // the significance of attribute $b \in B = A - red\prime(A)$ for $red\prime(A)$

Step 7: If $GD(red\prime(A) \cup b) = GD(A)$, then output $red(A) = red\prime(A) \cup b$, end.

If $GD(red\prime(A) \cup b) > GD(A)$, repeat step 6 until $GD(red\prime(A) \cup b) = GD(A)$, output $red(A)$, end.

Step 8: Calculate $Core(A) = \{a \in A | \, Sig_{A-\{a\}}(a) > 0\}$.

Step 9: If $GD(Core(A)) = GD(A)$, output $red(A) = Core(A)$, end.

If $GD(Core(A)) > GD(A)$, turn next;

Step10: Calculate $\max\limits_{b \in B} Sig_{Core(A)}(b)$, $B = A - Core(A)$ // the significance of attribute $b \in B = A - Core(A)$ for $Core(A)$

Step11: If $GD(Core(A) \cup b) = GD(A)$, output $red(A) = Core(A) \cup b$ and $Core(A)$, end.

If $GD(Core(A) \cup b) > GD(A)$, repeat step 10.

Step12: Calculate $\max\limits_{c \in C} Sig_{Core(A) \cup b}(b) \cup (Core(A) \cup b)$ // the significance of attribute $c \in C = A - Core(A) \cup b$ for $Core(A) \cup b$.

Step13: Output $red(A) = Core(A) \cup b \cup c \cup \cdots$ and $Core(A)$, end.

Example 2. Let $S = (U, A, V, f)$ be an information system (Table 2). There are 5 objects and 4 attributes, where $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = \{a_1, a_2, a_3, a_4\}$. Calculate the reduction of the system.

Table2. An information system

U	a_1	a_2	a_3	a_4
x_1	1	0	2	1
x_2	2	1	2	1
x_3	0	1	2	0
x_4	2	0	1	1
x_5	1	2	2	0

It is easy to calculate that
$U/A = \{x_1, x_2, x_3, x_4, x_5\}$, $GD(A) = \left(\sum\limits_{i=1}^{n} |X_i|^2\right) \Big/ |U|^2 = \frac{5}{25} = \frac{1}{5}$,
$U/A - \{a_1\} = \{x_1, x_2, x_3, x_4, x_5\}$, $GD(A - \{a_1\}) = \frac{1}{5}$,
$Sig_{A-\{a_1\}}(a_1) = GD(A - \{a_1\}) - GD(A) = 0$;
$U/A - \{a_2\} = \{x_1, x_2, x_3, x_4, x_5\}$, $GD(A - \{a_2\}) = \frac{1}{5}$,
$Sig_{A-\{a_2\}}(a_2) = GD(A - \{a_2\}) - GD(A) = 0$;
$U/A - \{a_3\} = \{x_1, x_2, x_3, x_4, x_5\}$, $GD(A - \{a_3\}) = \frac{1}{5}$,
$Sig_{A-\{a_3\}}(a_3) = GD(A - \{a_3\}) - GD(A) = 0$;
$U/A - \{a_4\} = \{x_1, x_2, x_3, x_4, x_5\}$, $GD(A - \{a_4\}) = \frac{1}{5}$,
$Sig_{A-\{a_4\}}(a_4) = GD(A - \{a_4\}) - GD(A) = 0$.
Based on the result above, we can see that the system has no core. Next we calculate the significance of attribute combination $Sig_{A-\{a_i,a_j\}}(a_i, a_j)$, $1 \le i \ne j \le 4$:
$GD(A - \{a_1, a_2\}) = \frac{9}{25}$, $Sig_{A-\{a_1,a_2\}}(a_1, a_2) = GD(A - \{a_1, a_2\}) - GD(A) = \frac{4}{25}$;
$GD(A - \{a_1, a_3\}) = \frac{7}{25}$, $Sig_{A-\{a_1,a_3\}}(a_1, a_3) = GD(A - \{a_1, a_3\}) - GD(A) = \frac{2}{25}$;
$GD(A - \{a_1, a_4\}) = \frac{7}{25}$, $Sig_{A-\{a_1,a_4\}}(a_1, a_4) = GD(A - \{a_1, a_4\}) - GD(A) = \frac{2}{25}$;
$GD(A - \{a_2, a_3\}) = \frac{5}{25}$, $Sig_{A-\{a_2,a_3\}}(a_2, a_3) = GD(A - \{a_2, a_3\}) - GD(A) = 0$;
$GD(A - \{a_2, a_4\}) = \frac{7}{25}$, $Sig_{A-\{a_2,a_4\}}(a_2, a_4) = GD(A - \{a_2, a_4\}) - GD(A) = \frac{2}{25}$;
$GD(A - \{a_3, a_4\}) = \frac{5}{25}$, $Sig_{A-\{a_3,a_4\}}(a_3, a_4) = GD(A - \{a_3, a_4\}) - GD(A) = 0$.
We can see that $GD(a_1, a_2) = GD(A)$, so $red(A) = \{a_1, a_2\}$ is the reduction of the system.

5 Results and Discussion

The idea of Algorithm 1 is listed as follow: At first, work out the reduction core $Core(A)$ by finding out the set $\{a \in A|\, Sig_{A-\{a\}}(a) > 0\}$. Then calculate the significance of the rest attribute relative to $Core(A)$. If $GD(Core(A) \cup a) = GD(A)$, then the set $Core(A) \cup a$ is the reduction of the information system. The time complexity of the algorithm is $T \approx O(|C|^3 \cdot |U|^2)$. The precondition of Algorithm 1 is working out reduction core. However, some information systems may have no reduction core. In this case, Algorithm 1 is powerless. In this paper, an improved algorithm is proposed. In an information system with no reduction core, the suboptimal reduction $red\prime(A)$ replaces the reduction core. The improved algorithm does not increase in time complexity. It can deal with the system with reduction core and no reduction core.

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A Retrospective Assessment of Fuzzy Logic Applications in Voice Communications and Speech Analytics

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Abstract: Voice and speech communication is a major topic covering simultaneously 'communication', 'control' (because it often involves control in the coding algorithms), and 'computing' - from speech analysis and recognition, to speech analytics and to speech coding over communication channels. While fuzzy logic was specifically conceived to deal with language and reasoning, it has yet a limited use in the referred field. We discuss some of the main current applications from the perspective of half a century since fuzzy logic inception.
Keywords: fuzzy logic, fuzzy system, speech, communication, VAD, speech segmentation, speech coding, speech analytics.

1 Introduction

At 50 years since the advent of fuzzy logic, 40 years since by Lotfi A. Zadeh introduced the concept of linguistic variable [50] and at more than 60 years since the mathematician Grigore C. Moisil argued that a new logic must be invented for describing human language and reasoning, it is compelling to ask: How much fuzzy logic did contributed to our understanding and technical use of language and speech in communications? As fuzzy logic (FL) was specifically conceived to model the human language and logic vagueness, one could expect that it played a central part in improving voice communications and speech recognition. Yet, the current situation does not seem to fully support this expectation – at least not at the level FL gained popularity in (fuzzy) control theory. In the review [44], not a single mention to fuzzy logic is made in connection to voice communication, showing no penetration in the mainstream of communication applications after almost 30 years since the first paper on FL. The situation has not much improved. The recent review [11] deplores the fact that there are very few if any papers using FL in the major conferences devoted to speech. Major journals publish only seldom papers on FL in speech applications and voice communications. As a matter of example, there is a single paper in this Journal to refer to the control and optimization of voice communications, rather indirectly, namely [16]. That paper does not use FL-based methods. There is also another paper referring to fuzzy control for data and indirectly, but not directly to voice communication, [48]. Sparsely, there are however papers on the topic in major journals, see for example the approach in [14] on a fuzzy traffic controller for ATM networks. An hypothesis for explaining this astonishing, general situation of the low number of papers on voice communication and speech is that something is still missing to allow for the expected eruption of FL applications in the field. We critically review the state of the art and search for answers for the current state of affairs. In this paper we assess the use of FL in four narrow sub-domains: voice activity detection (VAD), speech segmentation, and speech coding, which are strongly related, on one side, respectively speech analytics, on the other side.

2 Some Applications of FL to Voice Communications and Speech Coding

2.1 Fuzzy VADs and FL in Speech Segmentation

Voice communication in telephony systems and over Internet is done in digital form, by packets of data with speech coded using PCM or other coding techniques. It is a matter of minimizing the transmission effort and thus optimizing transmission capacity and minimizing energy to send only useful packets. Speech is full of pauses that may include ambient noise. Sending over the networks pause (noise) packets is useless. Therefore, detection of speech and noise (pause) segments, and next coding and sending only the speech segments may significantly improve communication efficiency and channel useful capacity. The so-called voice-activity detectors (VADs) are meant to separate useful and useless segments before transmission and are included in virtually all communication equipment. The main difficulty in building high quality VADs is to differentiate between consonants and noise, because some of the consonants, especially the fricative ones are noise-like. Both the frequency spectra and the amplitude of the fricatives as /s/, /f/ are close to white noise of low amplitude, as one encounters in offices. That makes difficult the task of the voice activity detectors. Compounded with that is the variability of the noise, which is typically nonstationary and may be white, pink, impulsive or a mixture of them, with variable amplitudes. Discerning between noise and unvoiced consonants is a matter of classification, possibly solved with fuzzy voice activity detection (FVAD) algorithms as in [3], [4], [5], [6], [7], [8], [12].

The VA detection needs several preliminary stages. In one approach, one detects and separates the periodical and a-periodical segments (PAP analysis) in the speech [3], [4], [5] using a linear predictor (LP). LPs approximate the sampled speech signal s_n as a linear combinations of the previous samples, according to $s_n^a = \sum_{k=0}^{M} a_k s_{n-k} + \sum_{j=1}^{Q} b_j s_{n-j}^a$, where s^a denotes the approximated samples and a_k, b_j are the LP coefficients. LPs are able to model well periodic signals, with low error $e_n = s_n - s_n^a$, while they are inefficient for a-periodic signals (large prediction errors). Signal parameters as the energy and the number of zero-crossings (NZC) are also used for supplementing the LP PAP analysis, where NZC is computed as the number of times, in a specified length segment of signal (signal window), successive samples satisfy $s_{n-1}s_n \leq 0$. Alternatively, one may use only amplitude, spectral properties such as the ratio of powers in the low and high frequency bands and NZC, possibly supplemented with the values of the self-correlation function, or properties of the cepstrum or of the Mel-spectrum (Mel-Frequency Cepstral Coefficients - MFCC) etc., to discriminate between voiced, unvoiced and noise segments. In VAD, as well as in speech segmentation and speech and emotion recognition, the decision is made based on the original parameters, such as LPC coefficients, energy, and NZC, or based on a set of derived, fused parameters – the representation space. In the second case, several parameters in the primary parameter space are processed together and a new representation (representation space) is derived, for example, the coherence between the periodic part of the signal and the noise (remaining part), as in [3], or the fuzzy information space representation as in [42].

VADs are included in communication standards, but none of the standards refers to FL and FVADs. Yet, [3] found that employing a decision based on FL rules applied to the 'coherence measure between the noisy speech and its prediction residue', the performances of the FVAD performs 'globally better than G.729B and presents moderate improvement when compared to UMTS 3G TS 26.094 VAD.' They used the coherence function computed on every frame k, $C(f,k)$, $C^2(f,k) = \frac{S_{sn}^2(f,k)}{S_s(f,k) \times S_n(f,k)}$, where f is the frequency, S denotes spectra in the frame for the noise $_n$ and signal $_s$, and S_{sn} stands for the inter-signal spectral density [3], [4]. After defining a set of frequency bands, B_i, the coherence function on each band is computed as

$C_i^B(k) = \sum_{f \in B_i} |C(f,k)|$ and these values are fuzzified according to three membership functions [3]. A fuzzy decision is optimized for determining the type of signal segment and thus the VA.

Beretelli et al. [6], [7], [8] tested another approach, using the same parameters employed by the ITU-T G.729 VAD standard, namely the energy differences between successive speech frames, for full-band ΔE_t and low-frequency band ΔE_L, the difference of the NZCs, ΔZC, and the spectral distortion ΔS between successive frames. In their VAD algorithm, the decision is made based on a set of simple fuzzy rules given in [7], such as 'IF (ΔS is medium or low) THEN (voice is active)' and 'IF (ΔE_L is low) AND (ΔS is very low) AND (ΔZC is high) THEN (voice is active)' (rules form [12]). Further improving the system, these authors considered multi-channel (two or several microphone) systems and took into account the delays between the signals. Using the output of the basic FVAD and the delays as inputs to a fuzzy network, after training the complex FVAD, and thus obtained better performing VADs than the simpler FVAD and than the G.729 standard VAD. Further refinements to increase the robustness in noise are given in [6], [7], [8], [12]. These authors report in [9] an improvement, compared with the VAD G.729, of more than 80% improvement in false activity detection.

A close topic is that of acoustic event detectors; we notice the interesting approach in [45], where information fusion for classification of non-speech sounds is performed by a skilled use of fuzzy integrals. Similarly, FL-based techniques applied to speech segmentation have been proposed by many authors, but the penetration of these techniques in the mainstream of speech segmentation is still limited. Speech segmentation may regard several levels, from voice activity to vowel (voiced sound)-consonant phoneme boundaries, to phonemic and syllabic unit segmentation, under various conditions of noise. Lin et al. [35] improved the noisy speech segmentation using neural fuzzy networks based on so-called 'adaptive time-frequency (ATF) and refined time-frequency (RTF) parameters'. Hsieh et al. [24] presented a neuro-fuzzy segmentation method specific for the Mandarin language, while [47] combined a context-dependent phonetic HMM recognizer with a fuzzy logic post-correction system that takes into account the conditions specific for each phonetic boundary for improving the precision of phoneme boundary determination. They report remarkable improvements, from errors of 400% for a basic HMM segmenter, compared to the durations determined by human operators, to a few percents after the corrections made by the fuzzy rules block.

There are three main classes of techniques for speech coding [44]: the waveform (direct) coding, coding in the model space, typically named parametric coding, and hybrid coding that is mixing the first two techniques. Speech coding is based on a compromise between speech perceived quality and the used bandwidth, and thus cost. The best quality is obtained by waveform coding methods, which are also the most costly in terms of transmitted bandwidth. Parametric (model) coding achieves low bandwidths, but the quality is poor-to-good at best.

Today speech coding, as in MPEG and telephony, is based on detailed psychoacoustic models derived from CELP. In brief, the low delay (LD) CELP coder standardized by CCITT as G.728 uses a 50^{th} order linear predictor (LP) excited by (i.e, having as input) predefined signals. The set of excitation signals is predefined and indexed on a 'codebook' (memory). After the LPC coefficients are determined on a speech frame, one searches the type of excitation and the best value of its amplitude (codebook gain) that produces at the LPC synthesizer output a signal that is the closest to the original speech frame (minimal error). The 'codebook' waveform index and the code of the best matching amplitude, together with the LPC coefficients code the speech frame. A perceptual filter is also used to improve the perceptual quality of the decoded speech signal. While the LP is computed with performant algorithms, the best choice of excitation and of its gain as available in the codebooks are time consuming. Sheikhan et al. [43] proposed a fuzzy adaptive resonance theory mapping (ARTMAP) for achieving fast codebook index selection. However, these authors have not justified their choice (the ARTMAP) in terms of the required computation power and time (complexity) and one could suspect that a simpler NN could have performed more efficiently in this application.

2.2 FL in Speech Analytics - A Surprising Low Development

Already in 2006 a Gartner report [18] found that audio search and speech analytics is one of the new technologies companies are adopting. Beyond marketing and services, speech analytics are used in various applications as security [49], learning and teaching [19]. Carlsson [10] argues that analytics and FL could be profitably combined in management.

There are multiple reasons to believe that FL may play an essential role both in interpreting the text and uncovering emotional states in speech; see for example the comparison of methods for emotion detection in [1], the example of method in [2] and the recent excellent paper [32]. However, many approaches applying FL to emotional speech are somewhat mechanistic, with no direct relevance for psychological, neurologic, and phonetic processes. There is virtually no FL or analytics-related study on the influence of the emotions on the articulatory processes (changes in vocal fold vibration and non-vocal fold vibration frequencies, degree of creakiness, changes in the articulation place and other elements of interest in articulatory phonetics). There are exceptions from the mechanistic approach to assessing the speaker state; such exceptions deserve recognition, e.g., [21], [22], [23], who study correlations between qualitative representations of emotions such as valence, activation, and dominance and the acousto-physical parameters (acoustic features). On the other hand, one has to recognize the market value of the mechanistic approaches in analytics and other applications: they aim to produce real-life applications such as synthesizing emotional speech for the games and movies industry [40], monitoring the state of drivers [26], [27], [28], call center control and crowd/social state monitoring and control. There was little research on differentiating simulated emotions with variable degrees of likeness to the true ones. Some studies addressed simulated emotions by non-actors, aiming to voice communication enhancement and education; e.g., [19], [38] found significant differences in emotion detection when comparing acted emotions by layman (corpus described in [19]) and actors. Genuine emotions, as studied in several other researches, were found more challenging to determine than the acted ones. We expect that FL can help represent the degree of likeness by actors and laymen, moreover help build emotion simulation detectors.

Because FL found reputed applications in classification, e.g., the kNN method and neuro-fuzzy classifiers [51] moreover because the analytics extensively use concepts easier interpretable by people than by current machines, one could expect that FL is largely present in analytics modules, including speech analytics. While this is not true, fuzzy ontologies are quite popular and at least some analytics have FL-based modules, see SAP-Hana [**?**] which considers fuzzy search as one of the 'few important techniques being used in Text Analysis'; namely fuzzy search stands for 'finding strings that match a pattern approximately'. Although this use relates to language, the technique simply applies FL in defining a fuzzy distance over the set of strings. The search is based on a minimal matching value and the respective command is like CONTAINS (<string-tolook-for>, FUZZY ($0.x$)), with $0.x$ the minimal accepted similarity [53]. Note that this analytics provides the function SCORE() that determines the degree of similarity for every string in a specified set and the given string, but this is simplistic and far from what would may be expected as level of use of FL in understanding people's language communication. An interesting direction was opened in [39], who proposed the use of prosodic features to prioritize the call servicing. While not using FL, the approach in [39] is an example of applications where FL looks promising for speech analytics. Similarly, there is an interesting paper, [13], thoroughly analyzing the possibilities of mining fuzzy association rules in texts; that track could be followed and applied on step further to finding fuzzy associations between textual information and prosody and emotions in speech. Another remarkable approach to analytics based on FL but not related to speech is constituted by a series of papers [30], [31] that apply fuzzy data analysis and inductive fuzzy classification using a normalization of the likelihood ratio to metadata and for knowledge discovery. Surprisingly, there are few reports on research on FL applied to speech analytics related to emotions. Maybe this is due to the fact that sentiment analysis on texts have developed earlier and that it is considered sufficient for deriving the mood of the speaker.

3 Discussion and Conclusions

While the contributions of FL to speech technology, specifically to VAD, speech segmentation, and coding cannot be disregarded, these contributions seem to be less significant than one may expect from applying FL to speech. Few researches compare the results and the advantages or disadvantages of the FL approach to non-fuzzy approaches, or even try to justify the FL-based approach. While some good results obtained using FL-approaches are expected based on the known power of universal approximation (and thus nonlinear classification) of FLSs, the capabilities of others, including their generalization power are less clear. A more systematic research program for employing FL in speech analysis is needed to overcome the current limits. An explanation for this state of affairs could be that FL requires extensive computations, while systems as cellular phones and even PCs are restricted in computation power. However, the recent processors have tremendously increased in computation power, favoring a larger use of FL. Thus, one can look forward with the hope that FL will achieve more in this field, in the near future.

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A Hybrid Model Based on Fuzzy AHP and Fuzzy WASPAS for Construction Site Selection

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Abstract: The purpose of this article is to propose a fuzzy multi-attribute performance measurement (MAPM) framework using the merits of both a novel Weighted Aggregated Sum-Product Assessment method with Fuzzy values (WASPAS-F) and Analytical Hierarchy Process (AHP). The object of this study is to select the best shopping centre construction site in Vilnius. A number of conflicting qualitative and quantitative attributes exist for evaluating alternative construction sites. Qualitative attributes are accompanied by ambiguities and vagueness. This makes fuzzy logic a more natural approach to this kind of multi-attribute decision making (MADM) problems. Fuzzy AHP is applied for assigning weights of the attributes and WASPAS-F method is used to determine the most suitable alternative.
Keywords: AHP, WASPAS-F, Fuzzy AHP, multi-attribute decision making (MADM), key performance attributes, construction site, shopping centre.

1 Introduction

The increasing competition cause a lot of construction site selection problems. Making decisions is a complex process that involves multiple, usually conflicting, objectives or attributes. They are ill-structured. Multi-criteria decision aid (MCDA, the European School) or multi-attribute decision making (MADM, the American School) constitutes an advanced field of operations research which is devoted to the development and implementation of decision support methodologies to confront complex decision problems. The problem of how decisions are or ought to be taken by individuals, organisations and institutions was previously discussed by Aristotle [1]. Later, in the 18th century Bernoulli [2] concentrated research on probability theory, Borda [3] on social choice procedures. The foundations of MCDA can be traced back in the works of von Neumann and Morgenstern [4] and Fishburn [5] on utility theory. Multi-attribute utility theory (MAUT) is an extension of the classical utility theory. The theory underlying multi-attribute performance measurement models was developed in the 1960s, as summarised in Keeney and Raiffa [6] and Zeleny [7].

Most of these methods have been developed based on the concepts of accurate measurements and crisp evaluation. The performance measurement parameters cannot be given precisely. The imprecision comes from different sources: unquantifiable information, incomplete information, no obtainable information, and partial ignorance. The real-world problems in performance measurement involve numerous aspects of uncertainty, contain a mixture of fuzzy and crisp data and may have a large number of alternatives and dozens of attributes. The solution is highly dependent on the preferences of the decision maker. It is common that people may not be 100% sure when making subjective judgments. If all or some of the alternatives are imprecise, then fuzzy MADM methods are required. Zimmermann's description of uncertainty is as follows:

Uncertainty implies that in a certain situation a person does not possess the information which quantitatively and qualitatively is appropriate to describe, prescribe or predict deterministically and numerically a system, its behaviour or other characteristics [8]. Fuzzy set theory is developed for solving problems, taking into account uncertainty, imprecision, vagueness. A pioneering and outstanding works on fuzzy sets are done by Zadeh [9] - [12]. The papers introduced a new perspective on the treatment of uncertainty, ambiguity, linguistic variables, and a fundamental aspect of formal languages: fuzzy set theory. Fuzzy measures can be introduced for two different uses: either they can represent a concept imprecisely known (although well defined) or a concept which is vaguely perceived such as in the case of a linguistic variable. The basic concepts and algorithms from classical MADM methods have been used in the development of the fuzzy MADM methods.
Fuzzy direct aggregation procedures of MADM methods generally consist of two stages:

1. The aggregation of the performance scores with respect to all attributes (goals) and per decision alternative, and

2. The rank ordering of the aggregated judgments of decision alternatives determine the optimal alternative.

Construction site selection is an important task. Proper construction site selection can improve project's success. Construction site selection for shopping centre requires a good visibility and accessibility, proper access to suppliers, customers, suitable transportation network and sufficient customer traffic and socio-economic population characteristics. The evaluation data of location performance of the construction site for various subjective attributes, and the weights of the attributes are usually expressed in linguistic terms. This makes fuzzy logic a more natural approach to this kind of problems. There is a limited number of papers evaluating suitable locations. Usually GIS in combination with decision making methods is applied for selecting the best site for ecologically and economically important objects such as renewable energy systems, including solar or wind farms [13] - [15]. Prioritizing the best sites for waste management is suggested by applying fuzzy TOPSIS [16] or VIKOR [17] methods. AHP is applied for warehouse [18] or solar farm [19] construction site selection. Also fuzzy MCDM frameworks for locating plants are suggested [20], [21]. Application of interval type-2 fuzzy sets for watershed site selection is presented [22]. Hybrid approach of fuzzy analytic network process (ANP), fuzzy DEMATEL and fuzzy ELECTRE for site selection is suggested [23].

The purpose of this article is to propose a novel fuzzy multi-attribute performance measurement framework using the merits of both a novel Weighted Aggregated Sum-Product Assessment method with Fuzzy values (WASPAS-F) and fuzzy Analytical Hierarchy Process (AHP).

2 Methodology

The high value of a problem-solving process is that it helps to align human's thinking and action around a common approach for winning team problem solving to the following nine steps:

1. Selecting the problem;

2. Exploring the problem and gathering data;

3. Establishing success attributes;

4. Developing a clear problem statement;

5. Generating alternatives;

6. Evaluating alternatives;

7. Selecting a preferred solution;

8. Developing a plan for action;

9. Testing and modifying the solution.

There is no evidence in the literature of publications evaluating the shopping centre construction sites were applied in Lithuania. This is the most powerful motivation to consider the site selection problem. Flowchart of proposed problem solving process is shown in Fig. 1. The first part of methodology, i.e. the fuzzy AHP was used to calculate attributes weights. In the second part, the WASPAS-F method was developed and used to rank and select the alternatives. First

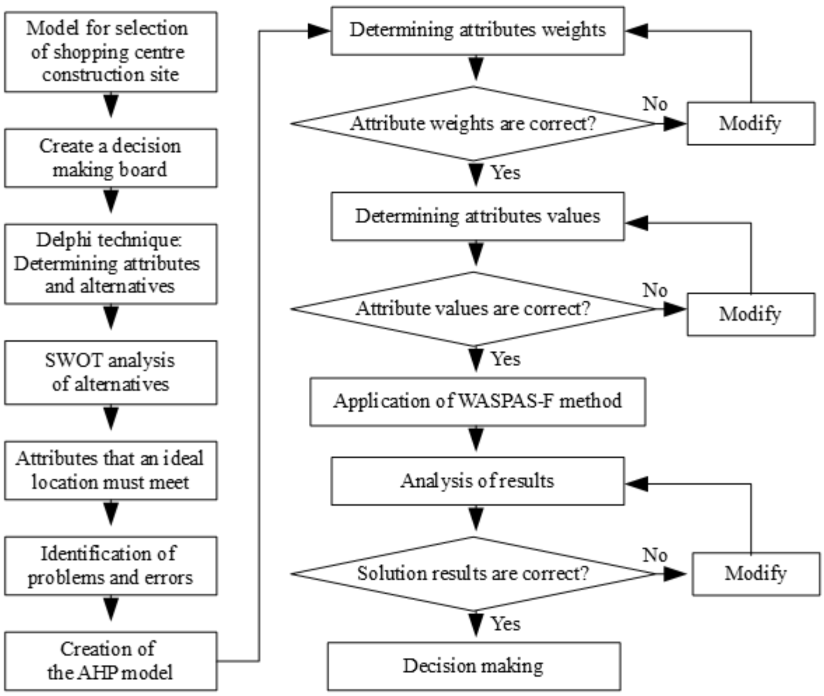

Figure 1: Flowchart of the proposed problem solving process

of all expert group was formed. The creditability of a group of experts depends on the expertise of the people who are involved and how they can give full play to their professional expertise and make wise and fair decisions. The group of experts was formed based on three main problem solving skills:

1. Communications patience (techniques that help members share information and perspectives by working hard to understand one another and working hard to be understood, members with controversial or divergent views are not ignored or blocked);

2. Synergy creation (techniques that equip members to expand their thinking by generating many ideas, building on those ideas, and evaluating ideas to create synergistic solutions);

3. Disciplined use of a problem solving process (members become disciplined in using a systematic process for analysing data, creating options, and evaluating and selecting preferred solutions).

In the absence of good data, members waste time in pointless debates over opinions, and the problem with opinions is that we all have different ones. The members were asked to generate as many ideas or options as they can without censorship or judgment. Ideas were evaluated considering all factors consequences and sequels of all the probable consequences of a particular scenario in the short term, the medium term, and the longer term. The members were asked to identify important criteria for assessing the feasibility of several ideas.

2.1 Preliminairies

A fuzzy set is a class of objects with a continuum of membership grades. Such set is characterized by a membership function which assigns to each object a grade of membership ranging between zero and one [9]. A fuzzy set A defined in space X is a set of pairs:

$$A = \{(x, \mu_A(x)), x \in X\}, \tag{1}$$

where the fuzzy set A is characterized by its membership function $\mu_A : X \to [0;1]$, which associates with each element $x \in X$, a real number $\mu_A(x) \in [0;1]$. The value $\mu_A(x)$ at x represents the grade of membership of x in A and is interpreted as the membership degree to which x belongs to A . So the closer the value $\mu_A(x)$ is to 1, the more x belongs to A.

A crisp or ordinary subset A of X can also be viewed as a fuzzy set in X with membership function as its characteristic function, i.e.

$$\mu_A(x) = \begin{cases} 1 & x \in A; \\ 0 & x \notin A. \end{cases} \tag{2}$$

The set X is called a universe of discourse and can be written $\subseteq X$. Sometimes a fuzzy set A in X is denoted by list the ordered pairs $(x, \mu_A(x))$, where the elements with zero degree are usually not listed. Thus a fuzzy set A in X can be represented as $A = \{(x, \mu_A(x))\}$, where $x \in X$ and $\mu_A : X \to [0;1]$.

When the universe of discourse is discrete and finite with cardinality n, that is $X = \{x_1, x_2, \ldots, x_n\}$, the fuzzy set A can be represented as

$$A = \sum_{i=1}^{n} \frac{\mu_A(x_i)}{x_i} = \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \ldots + \frac{\mu_A(x_n)}{x_n}, \tag{3}$$

when the universe of discourse X is an interval of real numbers, the fuzzy set A can be expressed as

$$A = \int_X \frac{\mu_A(x)}{x}. \tag{4}$$

A fuzzy number A is defined to be a fuzzy triangular number, with α - lower, β - modal, and γ -upper values, if its membership function $\mu_A : X \to [0;1]$ is fully described as follows:

$$\mu_A(x) = \begin{cases} \dfrac{x-\alpha}{\beta-\alpha} & \text{if } x \in [\alpha, \beta], \\ \dfrac{x-\alpha}{\beta-\gamma} & \text{if } x \in [\beta, \gamma], \\ 0 & \text{otherwise.} \end{cases} \quad \alpha \leqslant \beta \leqslant \gamma. \tag{5}$$

In order to obtain a crisp output, a defuzzification process is needed to be applied. Defuzzification is the process of producing a quantifiable result in fuzzy logic, given fuzzy sets and corresponding membership degrees. The output of the defuzzification process is a single number. Various types of membership functions are used. The most commonly used membership functions are the following [24]: triangular, trapezoid, linear, sigmoidal, π-type, and Gaussian.

A fuzzy number is generally a subjective data given by one expert or several (after collective agreement). The most typical fuzzy set membership function is triangular membership function (Fig. 2).

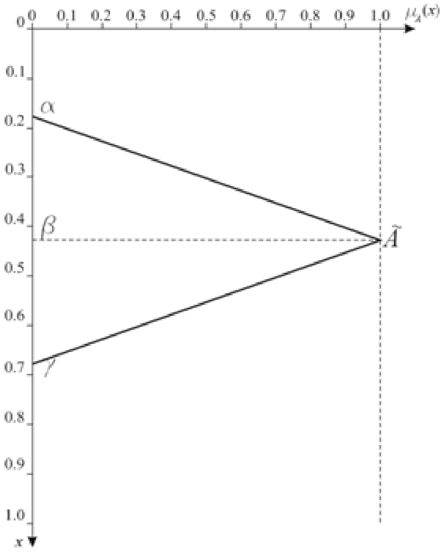

Figure 2: Triangular membership function

Van Laarhoven and Pedrycz [25] introduced the basic operations of fuzzy triangular numbers $\tilde{x}_1$ and $\tilde{x}_2$ (Table 1). In the Table 1 α is a lower value of fuzzy number, β - modal value of fuzzy number, γ -upper value of fuzzy number.

Values of weight $0 < \tilde{w}_j < 1, \sum_{j=1}^{n} \tilde{w}_j = 1$ are usually determined by experts. There are various approaches for assessing weights. Decision makers data which cannot be exactly described by means of numerical values, commonly describe the different ways they measure things numerically in terms of scales of measurement, which come in four flavours: nominal, ordinal, interval, or ratio scales. Likert items were first introduced in 1932 [26]. Likert scales can indeed be analysed effectively as interval or fuzzy scales [27] in which categories are labelled with numerical values.

2.2 Fuzzy AHP

The earliest work in fuzzy AHP appeared in van Laarhoven and Pedrycz [25], which compared fuzzy ratios described by triangular membership functions. Chang [28] introduced a new approach for handling fuzzy AHP, with the use of triangular fuzzy numbers for pairwise comparison scale of fuzzy AHP, and the use of the extent analysis method for the synthetic extent values of the pairwise comparisons. This study concentrates on a fuzzy AHP approach introduced by Chang [28], in which triangular fuzzy numbers are preferred for pairwise comparison scale. This questionnaire sought the satisfaction level of the experts concerning the candidate alternatives using a Likert-type five-point scale. The linguistic variables matching TFNs are provided in Fig.3.

Fuzzy group weight is determined as follows:
After obtaining the attributes weights from AHP the synthesising of ratio judgements is done. $\tilde{W} = [\overline{\tilde{w}_1, \tilde{w}_n}] = [\tilde{w}_j]$ fuzzy group weights for n attributes are determined as follows ($\tilde{w}_j$ is fuzzy

Table 1: Basic operations on fuzzy triangular numbers

Equation	Operation
$\tilde{x}_1 = (x_{1\alpha}, x_{1\beta}, x_{1\gamma}),\ \tilde{x}_2 = (x_{2\alpha}, x_{2\beta}, x_{2\gamma})$ (6)	fuzzy triangular numbers
$\tilde{x}_1 \oplus \tilde{x}_2 = \begin{pmatrix} x_{1\alpha}+x_{2\alpha}, \\ x_{1\beta}+x_{2\beta}, \\ x_{1\gamma}+x_{2\gamma} \end{pmatrix}$ (7)	addition
$\tilde{x}_1 \ominus \tilde{x}_2 = \begin{pmatrix} x_{1\alpha}-x_{2\gamma}, \\ x_{1\beta}-x_{2\beta}, \\ x_{1\gamma}-x_{2\alpha} \end{pmatrix}$ (8)	substraction
$\tilde{x}_1 \otimes \tilde{x}_2 = \begin{pmatrix} x_{1\alpha}x_{2\alpha}, \\ x_{1\beta}x_{2\beta}, \\ x_{1\gamma}x_{2\gamma} \end{pmatrix}$ (9)	multiplicaction
$k \otimes \tilde{x}_1 = \begin{pmatrix} kx_{1\alpha}, \\ kx_{1\beta}, \\ kx_{1\gamma} \end{pmatrix}$ (10)	multiplicaction by constant
$\tilde{x}_1 \div \tilde{x}_2 = \begin{pmatrix} x_{1\alpha}/x_{2\gamma}, \\ x_{1\beta}/x_{2\beta}, \\ x_{1\gamma}/x_{2\alpha} \end{pmatrix}$ (11)	division
$\tilde{x}_1^{-1} = \begin{pmatrix} 1/x_{1\gamma}, \\ 1/x_{1\beta}, \\ 1/x_{1\alpha} \end{pmatrix}$ (12)	reverse number
$\tilde{x}_1^{\tilde{x}_2} = \begin{pmatrix} x_{1\alpha}^{x_{2\gamma}}, \\ x_{1\beta}^{x_{2\beta}}, \\ x_{1\gamma}^{x_{2\alpha}} \end{pmatrix}$ (13)	raising a fuzzy triangular number of the power of another fuzzy triangular number, if $x_{1\alpha} \leqslant x_{1\beta} \leqslant x_{1\gamma} \leqslant 1$ and $x_{2\alpha} \leqslant x_{2\beta} \leqslant x_{2\gamma} \leqslant 1$ is special for this case study

triangular number):

$$\tilde{x}_j = (x_{j\alpha}, x_{j\beta}, x_{j\gamma}), \tag{14}$$

where y_{jk} is j attribute weight determined by k expert, p is number of experts, $w_{j\alpha} = \min_k y_{jk}, j = \overline{1,n}, k = \overline{1,p}$ is minimum possible value, $w_{j\beta} = \left(\prod_{k=1}^{p} y_{jk}\right)^{1/p}, j = \overline{1,n}$ is the most possible value and $w_{j\gamma} = \max_k y_{jk}, j = \overline{1,n}, k = \overline{1,p}$ is maximum possible value of j attribute weight.

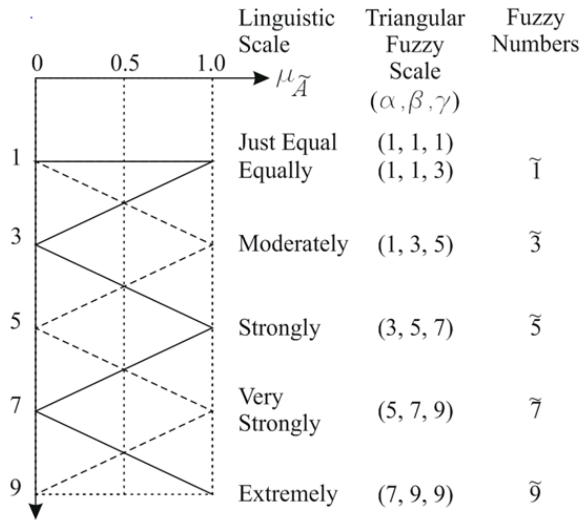

Figure 3: Linguistic variables for the importance weight of each attribute.

2.3 A novel fuzzy multi-attribute performance measurement model: Weighted Aggregated Sum-Product Assessment method with Fuzzy values (WASPAS-F)

This subsection extends WASPAS to the fuzzy environment. The merit of using a fuzzy approach is to assign the relative importance of attributes using fuzzy numbers instead of precise numbers. MADM method, namely WASPAS, was introduced in 2012 by Zavadskas et al. [29]. Later, modification of the method WASPAS-IFIV was introduced [30]. There are a number of applications of WASPAS method, including site selection for wind turbine [31], deciding shopping mall locating [32] or assessing sites for implementation of solar projects [33]. Also the method was successfully applied for evaluating alternative technological or design solutions in construction [34] - [37], manufacturing [38], business issues [39] or even for performance analysis and ranking of scholarly journals [40].
The WASPAS method consists of two aggregated parts:

1. The Weighted Sum Model (WSM);

2. The Weighted Product Model (WPM).

The WSM method is simple, easy to use and understood. It determines overall score of an alternative as a weighted sum of the attribute values. It is the best known and most widely used method [41]. The WPM is developed in order to avoid alternatives with poor attribute values. It determines score of each alternative as a product of the scale rating of each attribute to a power equal to the importance weight of the attribute [42].

Based on the briefly summarized fuzzy theory above, WASPAS-F steps can be outlined as follows:

Step 1. Forming of fuzzy decision-making matrix (FDMM). The performance values $\tilde{x}_{ij}$ and the attributes weights $\tilde{w}_j$ are entries of a DMM. Choose the linguistic ratings.

The system of attributes as well as the values and initial weights of attributes are determined by experts. The discrete optimization problem is represented by the preferences for m reasonable alternatives (rows) rated on n attributes (columns):

$$\tilde{X} = \begin{bmatrix} \tilde{x}_{11} & \dots & \tilde{x}_{1j} & \dots & \tilde{x}_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \tilde{x}_{i1} & \dots & \tilde{x}_{ij} & \dots & \tilde{x}_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \tilde{x}_{m1} & \dots & \tilde{x}_{mj} & \dots & \tilde{x}_{mn} \end{bmatrix}; i = \overline{1,m}, j = \overline{1,n}, \tag{15}$$

where $\tilde{x}_{ij}$ – fuzzy value representing the performance value of the i alternative in terms of the j attribute. A tilde ˜ is placed above a symbol if the symbol represents a fuzzy set.

Then the determination of the priorities of alternatives is carried out in several steps.

Step 2. The initial values of all the attributes $\tilde{x}_{ij}$ are normalised – defining values $\tilde{\bar{x}}_{ij}$ of normalised decision-making matrix $\tilde{\bar{X}} = [\tilde{\bar{x}}_{ij}]_{m \times n}$.

$$\tilde{\bar{x}}_{ij} = \begin{cases} \dfrac{\tilde{x}_{ij}}{\max\limits_i \tilde{x}_{ij}} & \text{if } \max\limits_i \tilde{x}_{ij} \text{ is preferable,} \\ \dfrac{\min\limits_i \tilde{x}_{ij}}{\tilde{x}_{ij}} & \text{if } \min\limits_i \tilde{x}_{ij} \text{ is preferable;} \end{cases} \quad i = \overline{1,m}, j = \overline{1,n}. \tag{16}$$

Step 3a. Calculate the weighted normalised fuzzy decision matrix $\tilde{\hat{X}}_q$ for WSM:

$$\tilde{\hat{X}}_q = \begin{bmatrix} \tilde{\hat{x}}_{11} & \dots & \tilde{\hat{x}}_{1j} & \dots & \tilde{\hat{x}}_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \tilde{\hat{x}}_{i1} & \dots & \tilde{\hat{x}}_{ij} & \dots & \tilde{\hat{x}}_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \tilde{\hat{x}}_{m1} & \dots & \tilde{\hat{x}}_{mj} & \dots & \tilde{\hat{x}}_{mn} \end{bmatrix}; \tilde{\hat{x}}_{ij} = \tilde{\bar{x}}_{ij}\tilde{w}_j, i = \overline{1,m}, j = \overline{1,n}. \tag{17}$$

Step 3b. Calculate the weighted normalised fuzzy decision matrix $\tilde{\hat{X}}_p$ for WPM:

$$\tilde{\hat{X}}_p = \begin{bmatrix} \tilde{\hat{\bar{x}}}_{11} & \dots & \tilde{\hat{\bar{x}}}_{1j} & \dots & \tilde{\hat{\bar{x}}}_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \tilde{\hat{\bar{x}}}_{i1} & \dots & \tilde{\hat{\bar{x}}}_{ij} & \dots & \tilde{\hat{\bar{x}}}_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \tilde{\hat{\bar{x}}}_{m1} & \dots & \tilde{\hat{\bar{x}}}_{mj} & \dots & \tilde{\hat{\bar{x}}}_{mn} \end{bmatrix}; \tilde{\hat{\bar{x}}}_{ij} = \tilde{\bar{x}}_{ij}^{\tilde{w}_j}, i = \overline{1,m}, j = \overline{1,n}. \tag{18}$$

Step 4. Calculate values of the optimality function:

a) according to the WSM for each alternative:

$$\tilde{Q}_i = \sum_{j=1}^{n} \tilde{\hat{x}}_{ij}, i = \overline{1,m}, \tag{19}$$

b) according to the WPM for each alternative:

$$\tilde{P}_i = \prod_{j=1}^{n} \tilde{\hat{\bar{x}}}_{ij}, i = \overline{1,m}. \tag{20}$$

The result of fuzzy performance measurement for each alternative are fuzzy numbers $\tilde{Q}_i$ and $\tilde{P}_i$. The centre-of-area is the most practical and simple to apply for defuzzification:

$$Q_i = \frac{1}{3}(Q_{i\alpha} + Q_{i\beta} + Q_{i\gamma}). \tag{21}$$

$$P_i = \frac{1}{3}(P_{i\alpha} + P_{i\beta} + P_{i\gamma}). \tag{22}$$

Step 5. The integrated utility function value of the WASPAS-F method for an alternative could be determined as follows:

$$K_i = \lambda \sum_{j=1}^{m} Q_i + (1-\lambda) \sum_{j=1}^{m} P_i, \lambda = 0, ..., 1, 0 \leqslant K_i \leqslant 1. \tag{23}$$

λ is determined based on the assumption that total of all alternatives WSM scores must be equal to the total of WPM scores:

$$\lambda = \frac{\sum\limits_{i=1}^{m} P_i}{\sum\limits_{i=1}^{m} Q_i + \sum\limits_{i=1}^{m} P_i}. \tag{24}$$

Step 6. Rank preference order. Choose an alternative with maximal K_i value.

3 Construction site for shopping centre site selection in Vilnius

In this case study, an investor company conducted a feasibility study to establish strategies to locate an appropriate shopping centre in strategic demand areas of Vilnius. The identification of feasible sites and the selection of the one will optimize the company's performance strategically. The problem was solved according to the presented above methodology. The MAPM model has been designed by considering the interests and objectives of all stakeholders, customers, local community, employees and suppliers. First of all, expert's team determined the potential critical errors of construction site location selection (Fig. 4). Then, based on expert team questionnaire they determined the main problems of construction site selection (Fig. 5).

Figure 4: Critical errors of construction site selection.

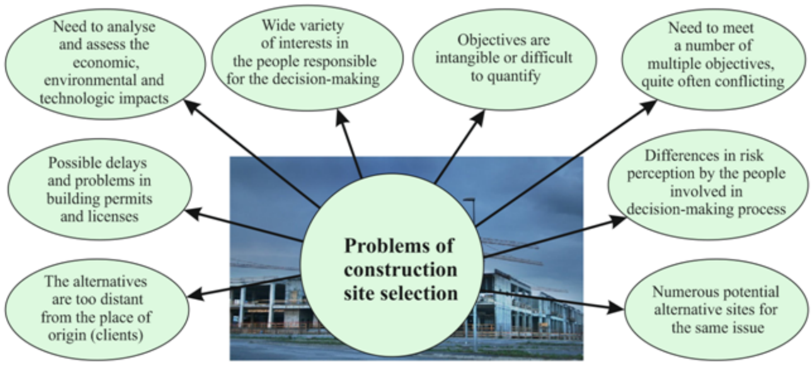

Figure 5: Problems of construction site selection.

An interview to evaluate the most suitable location alternatives was conducted with seven experts, including stakeholders, business development managers, real estate consultants and academicians. The next step was collection of the information associated to all of the alternative construction sites identified and a description of the most important attributes. The Strengths, Weaknesses, Opportunities, and Threats analysis was performed. Finally, four location alternatives, which are denoted as A_1, A_2, A_3 and A_4 are selected as feasible alternatives. A_1 alternative is at the intersection of important roads between the airport and the railway station and the central bus station. A_2 is one of the developing dwelling districts. A_3 is the most populated residential district. A_4 is one of the nearby the centre point of the Vilnius.

Later, the attributes were compared each other using AHP. A Likert-type ten-point scale (Fig. 3) was used.

In the next step the questionnaire was about experts' satisfaction level toward construction site selection. This questionnaire adopts a Likert-type ten-point scale (Fig. 6). It has ten different levels - from "very bad" ($\tilde{P}_1$) to "excellent" ($\tilde{P}_{10}$) - on a fuzzy ten-level scale. For example, expert might think the satisfaction the linguistic score of "fair" ($\tilde{P}_5$), that score would or correspond to a TFN of $(0.4, 0.5, 0.6)$ respectively. The pairwise comparison matrix set by TFNs that matches linguistic statements of data is shown in Table 2. Each of experts determined attributes weights. Integrated results of established weights are shown in Table 3. The priority weight vector describes the importance degree of the attributes in decision matrix. After getting the importance degree of attributes, WASPAS-F method was employed to evaluate alternative locations. In this phase of the study, WASPAS-F starts establishing fuzzy evaluations of the alternative locations (A_1, A_2, A_3 and A_4) with respect to the attributes by using TFNs. This is an initial decision making matrix for ranking alternatives and indicates the performance ratings of the alternatives according to the attributes. The alternatives according to linguistic scales and their corresponding fuzzy numbers $\{(1, 1, 1)$ – very poor, $(2, 3, 4)$ – poor, $(4, 5, 6)$ – fair, $(6, 7, 8)$ – good, $(8, 9, 10)$ – very good$\}$ are compared by experts. Table 4 shows comparison of alternatives according to attributes. The normalised decision matrix is obtained by using Eq. (16) (Table 5). The weighted normalised fuzzy decision matrix for WSM is obtained by using Eq. (17) (Table 6) and one for WPM is obtained by using Eq. (18) (Table 7). Values of the optimality function values for WSM and WPM are calculated by using Eq. (19) and Eq. (20) respectively. The integrated utility function value of the WASPAS-F method for an alternative was determined by using Eq. (24), as presented in Table 8.

As can be seen from Table 8, A_3 is the best alternative in the WSM, WPM and WASPAS method. A_4 alternative is the worst among considered alternatives. Decision maker should choose and implement the A_3 alternative.

Table 2: Pairwise comparisons of site selection attributes for shopping centre via TFN (The first Expert).

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	w
Construction costs	x_1	1, 1, 1	0.20, 0.33, 1	1, 1, 3	1, 3, 5	5, 7, 9	3, 5, 7	3, 5, 7	3, 5, 7	0.21
Economical	x_2	1, 3, 5	1, 1, 1	1, 1, 3	1, 3, 5	3, 5, 7	1, 3, 5	3, 5, 7	3, 5, 7	0.21
Road accesses	x_3	0.33, 1, 1	0.33, 1, 1	1, 1, 1	0.20, 0.33, 1	1, 3, 5	3, 5, 7	1, 3, 5	1, 3, 5	0.17
Competition	x_4	0.20, 0.33, 1	0.20, 0.33, 1	1, 3, 5	1, 1, 1	1, 3, 5	1, 3, 5	1, 3, 5	1, 3, 5	0.17
Population characteristics	x_5	0.11, 0.14, 0.20	0.14, 0.20, 0.33	0.20, 0.33, 1	0.20, 0.33, 1	1, 1, 1	3, 5, 7	1, 3, 5	1, 1, 3	0.12
Environmental impacts	x_6	0.14, 0.20, 0.33	0.20, 0.33, 1	0.14, 0.20, 0.33	0.20, 0.33, 1	0.14, 0.20, 0.33	1, 1, 1	1, 3, 5	1, 1, 3	0.07
Risks	x_7	0.14, 0.20, 0.33	0.14, 0.20, 0.33	0.20, 0.33, 1	0.20, 0.33, 1	0.20, 0.33, 1	0.20, 0.33, 1	1, 1, 1	1, 1, 3	0.04
Attractiveness	x_8	0.14, 0.20, 0.33	0.14, 0.20, 0.33	0.20, 0.33, 1	0.20, 0.33, 1	0.33, 1, 1	0.33, 1, 1	0.33, 1, 1	1, 1, 1	0.01

Table 3: Fuzzy weights of attributes

	Experts							$\tilde{W}$		
	E_1	E_2	E_3	E_4	E_5	E_6	E_7	$w_{j\alpha}$	$w_{j\beta}$	$w_{j\gamma}$
x_1	0.21	0.33	0.29	0.31	0.23	0.35	0.27	0.21	0.28	0.35
x_2	0.21	0.23	0.18	0.21	0.23	0.17	0.16	0.16	0.20	0.23
x_3	0.17	0.16	0.14	0.17	0.14	0.17	0.16	0.14	0.16	0.17
x_4	0.17	0.11	0.09	0.10	0.14	0.12	0.12	0.09	0.12	0.17
x_5	0.12	0.07	0.09	0.07	0.08	0.08	0.09	0.07	0.08	0.12
x_6	0.07	0.05	0.07	0.05	0.08	0.05	0.09	0.05	0.06	0.09
x_7	0.04	0.03	0.07	0.05	0.05	0.04	0.05	0.03	0.05	0.07
x_8	0.01	0.02	0.06	0.02	0.05	0.02	0.05	0.01	0.03	0.06

Table 4: The initial fuzzy decision making matrix for construction site selection

	W			A_1			A_2			A_3			A_4			max
	α	β	γ	α	β	γ	α	β	γ	α	β	γ	α	β	γ	
$\tilde{x}_1$	0.21	0.28	0.35	0.5	0.6	0.7	0.6	0.7	0.8	0.6	0.7	0.8	0.6	0.7	0.8	0.8
$\tilde{x}_2$	0.16	0.20	0.23	0.6	0.7	0.8	0.6	0.7	0.8	0.8	0.9	1.0	0.5	0.6	0.7	1
$\tilde{x}_3$	0.14	0.16	0.17	0.8	0.9	1.0	0.5	0.6	0.7	0.6	0.7	0.8	0.6	0.7	0.8	1
$\tilde{x}_4$	0.09	0.12	0.17	0.5	0.6	0.7	0.6	0.7	0.8	0.5	0.6	0.7	0.4	0.5	0.6	0.8
$\tilde{x}_5$	0.07	0.08	0.12	0.8	0.9	1.0	0.7	0.8	0.9	0.6	0.7	0.8	0.5	0.6	0.7	1
$\tilde{x}_6$	0.05	0.06	0.09	0.5	0.6	0.7	0.8	0.9	1.0	0.6	0.7	0.8	0.8	0.9	1.0	1
$\tilde{x}_7$	0.03	0.05	0.07	0.4	0.5	0.6	0.5	0.6	0.7	0.8	0.9	1.0	0.7	0.8	0.9	1
$\tilde{x}_8$	0.01	0.03	0.06	0.5	0.6	0.7	0.4	0.5	0.6	0.4	0.5	0.6	0.5	0.6	0.7	0.7

Table 5: The normalised fuzzy decision making matrix

	$\tilde{W}$			A_1			A_2			A_3			A_4		
	α	β	γ	α	β	γ	α	β	γ	α	β	γ	α	β	γ
$\tilde{\tilde{x}}_1$	0.21	0.28	0.35	0.63	0.75	0.88	0.75	0.88	1.00	0.75	0.88	1.00	0.75	0.88	1.00
$\tilde{\tilde{x}}_2$	0.16	0.20	0.23	0.60	0.70	0.80	0.60	0.70	0.80	0.80	0.90	1.00	0.50	0.60	0.70
$\tilde{\tilde{x}}_3$	0.14	0.16	0.17	0.80	0.90	1.00	0.50	0.60	0.70	0.60	0.70	0.80	0.60	0.70	0.80
$\tilde{\tilde{x}}_4$	0.09	0.12	0.17	0.63	0.75	0.88	0.75	0.88	1.00	0.63	0.75	0.88	0.50	0.63	0.75
$\tilde{\tilde{x}}_5$	0.07	0.08	0.12	0.80	0.90	1.00	0.70	0.80	0.90	0.60	0.70	0.80	0.50	0.60	0.70
$\tilde{\tilde{x}}_6$	0.05	0.06	0.09	0.50	0.60	0.70	0.80	0.90	1.00	0.60	0.70	0.80	0.80	0.90	1.00
$\tilde{\tilde{x}}_7$	0.03	0.05	0.07	0.40	0.50	0.60	0.50	0.60	0.70	0.80	0.90	1.00	0.70	0.80	0.90
$\tilde{\tilde{x}}_8$	0.01	0.03	0.06	0.71	0.86	1.00	0.57	0.71	0.86	0.57	0.71	0.86	0.71	0.86	1.00

Table 6: The weighted normalised matrix for WSM

	A_1			A_2			A_3			A_4		
	α	β	γ	α	β	γ	α	β	γ	α	β	γ
$\tilde{\hat{x}}_1$	0.13	0.21	0.31	0.16	0.25	0.35	0.16	0.25	0.35	0.16	0.25	0.35
$\tilde{\hat{x}}_2$	0.10	0.14	0.18	0.10	0.14	0.18	0.13	0.18	0.23	0.08	0.12	0.16
$\tilde{\hat{x}}_3$	0.11	0.14	0.17	0.07	0.10	0.12	0.08	0.11	0.14	0.08	0.11	0.14
$\tilde{\hat{x}}_4$	0.06	0.09	0.15	0.07	0.11	0.17	0.06	0.09	0.15	0.05	0.08	0.13
$\tilde{\hat{x}}_5$	0.06	0.07	0.12	0.05	0.06	0.11	0.04	0.06	0.10	0.04	0.05	0.08
$\tilde{\hat{x}}_6$	0.03	0.04	0.06	0.04	0.05	0.09	0.03	0.04	0.07	0.04	0.05	0.09
$\tilde{\hat{x}}_7$	0.01	0.03	0.04	0.02	0.03	0.05	0.02	0.05	0.07	0.02	0.04	0.06
$\tilde{\hat{x}}_8$	0.01	0.03	0.06	0.01	0.02	0.05	0.01	0.02	0.05	0.01	0.03	0.06
Q			0.78			0.79			0.82			0.75
$\sum Q_i$												3.15

Table 7: The weighted normalised matrix for WPM

	A_1			A_2			A_3			A_4		
	α	β	γ	α	β	γ	α	β	γ	α	β	γ
$\tilde{\bar{\bar{x}}}_1$	0.85	0.92	0.97	0.90	0.96	1.00	0.90	0.96	1.00	0.90	0.96	1.00
$\tilde{\bar{\bar{x}}}_2$	0.89	0.93	0.96	0.89	0.93	0.96	0.95	0.98	1.00	0.85	0.90	0.94
$\tilde{\bar{\bar{x}}}_3$	0.96	0.98	1.00	0.89	0.92	0.95	0.92	0.94	0.97	0.92	0.94	0.97
$\tilde{\bar{\bar{x}}}_4$	0.92	0.97	0.99	0.95	0.98	1.00	0.92	0.97	0.99	0.89	0.95	0.97
$\tilde{\bar{\bar{x}}}_5$	0.97	0.99	1.00	0.96	0.98	0.99	0.94	0.97	0.98	0.92	0.96	0.98
$\tilde{\bar{\bar{x}}}_6$	0.94	0.97	0.98	0.98	0.99	1.00	0.96	0.98	0.99	0.98	0.99	1.00
$\tilde{\bar{\bar{x}}}_7$	0.94	0.97	0.98	0.95	0.97	0.99	0.98	0.99	1.00	0.98	0.99	1.00
$\tilde{\bar{\bar{x}}}_8$	0.98	1.00	1.00	0.97	0.99	1.00	0.97	0.99	1.00	0.98	1.00	1.00
P			0.74			0.75			0.79			0.71
$\sum P_i$												2.99

Table 8: Integrated utility function values of the WASPAS-F method

	A_1	A_2	A_3	A_4
Q	0.78	0.79	0.82	0.75
P	0.74	0.75	0.79	0.71
λ	0.49			
K	0.76	0.77	0.80	0.73
Rank	3	2	1	4

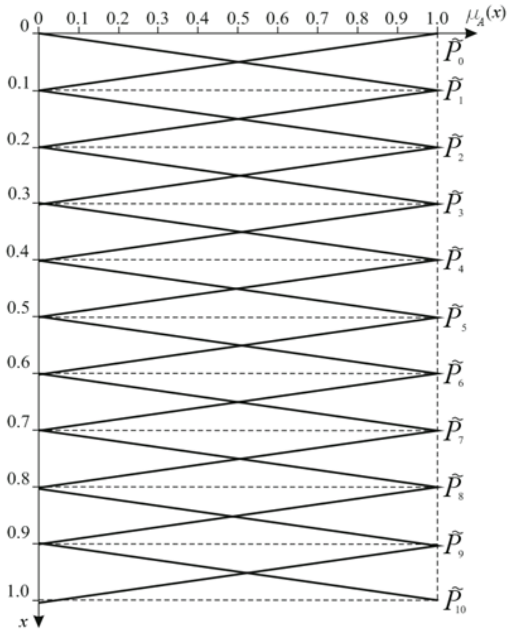

Figure 6: Membership functions of linguistic values for criteria rating (Likert-type ten-point scale).

4 Conclusions

Today, increase in population well-being and income causes the need of new construction sites. Selecting the best location for a new construction site in fuzzy environments becomes a difficult task for stakeholders. This paper proposed a combined fuzzy MADM approach based on the fuzzy AHP and WASPAS-F methods for selecting a suitable construction site location. In the proposed method, the fuzzy AHP was used to determine the weights of the attributes, while WASPAS-F was employed to rank the alternative locations. The approach combines the strong sides of the AHP and WASPAS methods. As a result of the study, we find that the proposed method is practical for ranking alternatives with respect to multiple conflicting attributes for the large scale problems.

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Homeomorphism Problems of Fuzzy Real Number Space and The Space of Bounded Functions with Same Monotonicity on [-1,1]

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Abstract: In this paper, based on the fuzzy structured element, we prove that there is a bijection function between the fuzzy number space ε^1 and the space $B[-1,1]$, which defined as a set of standard monotonic bounded functions with monotonicity on interval $[-1,1]$. Furthermore, a new approach based upon the monotonic bounded functions has been proposed to create fuzzy numbers and represent them by suing fuzzy structured element. In order to make two different metrics based space in $B[-1,1]$, Hausdorff metric and L_p metric, which both are classical functional metrics, are adopted and their topological properties are discussed. In addition, by the means of introducing fuzzy functional to space $B[-1,1]$, we present two new fuzzy number's metrics. Finally, according to the proof of homeomorphism between fuzzy number space ε^1 and the space $B[-1,1]$, it's argued that not only does it give a new way to study the fuzzy analysis theory, but also makes the study of fuzzy number space easier.

Keywords: fuzzy numbers; fuzzy structured element, standard monotonic bounded functions, fuzzy functional, homeomorphism

1 Introduction

Fuzzy numbers, which are a generalization of real numbers, have been perfectly applied to model and show the fuzzy data. Recently, application of fuzzy numbers in data mining algorithms has been an interesting topic to the researchers in this domain, for instance, clustering [1, 2], classification [3] and regression [4, 5]. Generally, the efforts have been done in study of fuzzy mathematical analysis and its application falls into two main categories:

First, studies on constructing fuzzy number metrics based on the fuzzy numbers and their topological properties. Many researchers proposed different metrics and many discussions on them have been proposed. For example, Hausdorff metric [6], L_p metric [7] and sendograph metric [8], were proposed as some of the most well-known widely used metrics.

The second category consists of those studies which addresses the relationship between the fuzzy number space and other topological spaces, study the properties of the fuzzy number space and develop some new methods in the proposed spaces. Among these studies, Goetschel and Voxman(2003) introduced a homeomorphic mapping from θ-crisp fuzzy number space to Hilbert space ℓ_2, which ranges in a convex cone (see [9]). Later, Gerg [10] generalized this mapping by extending the θ-crisp fuzzy number space to a more general one.

In order to apply the functional analysis to the fuzzy-valued functions studies, in which variables are real numbers and function values are fuzzy numbers, Puri and Ralescu [11], proposed an embedding theorem that the fuzzy number space ε^1 can be embedded into a Banach space X, with the help of the Radstrom embedding theorem of compact convex set. This theorem establishes the theoretical link between the fuzzy number space and the Banach space. However, because of do not considering any specific structure of Banach space, it is not easy to implement (it is not applicable anymore). Thus, by adopting the mapping of Goetschel and Voxman, Wu and Ma [12, 13] embedded fuzzy number space into the concrete Banach space $C[0, 1] \times C[0, 1]$ ($C[0, 1] = \{f : f$ is a bounded left-continuous function on $(0, 1]$, and f has right limit on $(0, 1]$, especially f is right-continuous at $0\}$), and present a specific isometrically isomorphic operator. Although the proposed embedding operator is proved to be as same as the embedding operator given by Puri and Ralescu [11] in the sense of isometrical isomorphism, the embedding operator has a specific form.

This paper is organized as follows: Section 2 and Section 3 introduced definitions and notations employed throughout the paper. In section 4, we introduced a specific fuzzy number, namely; fuzzy structured element. Then two important theorems which are the local mapping theorem and the structured element representation theorem of fuzzy number, are proved. As a result, we obtain a conclusion that there exist an one-one mapping from $B[-1, 1]$ with the same order standard monotonic bounded function family on $[0, 1]$ to the fuzzy real number space ε^1. In section 5, we introduced Lp metric and Hausdorff metric into $B[-1, 1]$ and some of its topological properties, such as completeness and separability are discussed. In section 6, by means of a fuzzy functional induced by fuzzy structured element, two fuzzy number metrics induced in the given metrics of $B[-1, 1]$. This section discussed the homeomorphism Problems between $B[-1, 1]$ and the space ε^1. Finally, we conclude and provide future works in Section 7.

2 Notion of the fuzzy numbers

Fuzzy numbers are the natural generalization of real and crisp numbers. A fuzzy number is a set of the real line with the upper semi-continuous and quasi-concave membership function. The definition implies that α-cut (A_α) of a fuzzy subset A is a closed interval in $[A_l^\alpha, A_r^\alpha]$ for any $\alpha \in (0, 1]$. The support of a fuzzy number A is a crisp set so that $\mathrm{supp}A = \mathrm{cl}(\{x : A(x) > 0\}) = [A_l^0, A_r^0]$(the closure of the support of A). Thus, by supposing $\mathrm{supp}A$ to be a bounded closed interval, A defined as a bounded fuzzy number. Denote all bounded fuzzy numbers on real line $\mathbb{R}$ as $\tilde{N}_c(\mathbb{R})$(or ε^1).

Theorem 1. *[12] If $u \in \tilde{N}_c(\mathbb{R})$, then let*

$$\underline{u}(\alpha) = \inf\{x : x \in u_\alpha\}, \quad \overline{u}(\alpha) = \sup\{x : x \in u_\alpha\},$$

here $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$ are two functions that satisfy the following conditions (1)–(4) on $[0,1]$:

(1) $\underline{u}(\alpha)$ is a bounded left continuous nondecreasing function on $(0,1]$;

(2) $\overline{u}(\alpha)$ is a bounded left continuous nonincreasing function on $(0,1]$;

(3) $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$ are right continuous at $\alpha = 0$;

(4) $\underline{u}(\alpha) \leq \overline{u}(\alpha)$.

Conversely, if functions $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$ satisfy the conditions (1)–(4) on $[0,1]$, then there exists an unique $u \in \tilde{N}_c(R)$ such that $u_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)]$ for each $\alpha \in [0,1]$.

The theorem says that for any fuzzy number A, it can be uniquely determined by two monotonic functions $\underline{u}(\alpha), \overline{u}(\alpha)$ on interval $[0,1]$.

3 Notions of the Extended set-valued function and general inverse function

Let f be a monotonic and bounded function on $[a,b]$ and $x_0 \in (a,b)$ be a discontinuous point in f. By considering f as a monotone increasing function, f can be a surjective function from $[a,b]$ to $(-\infty,+\infty)$ by the following formula:

$$f(x_0) = [f(x_0-), f(x_0+)], f(a) = (-\infty, f(a+)], f(b) = [f(b-), +\infty),$$

Here, we denote a new function $\hat{f}$, which $\hat{f}$ is a monotonic set-valued function extended by f and it also called extensional set-valued function of f. Furthermore, we denote all the family of function f which are bounded and have the same monotonicity on $[a,b]$, by $D[a,b]$.

3.1 Discontinue monotonic function with set-valued extensional at discontinuity

For discontinue monotonic increasing function f, x_0 is a discontinuous point in the range of $[-1,1]$. Here, $f(x_0 - 0) = m_1$ and $f(x_0 + 0) = m_2$, by considering our default suppose that f is an increasing function, then $m_1 < m_2$ and $f(x_0)$ is an interval number between $[m_1, m_2]$. If functional values of all discontinuities redefined as closed interval with left-hand and right-hand limited values, then this new function is called monotonic bounded set-valued function extensional from f that we denote it by $\hat{f}$. Obviously, inverse function $\hat{f}^{-1}$ of $\hat{f}$ exist.

3.2 Continuous non-strictly monotonic function

Suppose f is a non-strictly increasing function, then there exists at least one pair points $\{x_1, x_2\}$ on $[-1.1]$ such that value of f is equal to constant $c = f(x_1) = f(x_2)$ on interval $[x_1, x_2]$. And suppose x_1, x_2 are two endpoints so that increasing function f is equal to constant, that is, when $x < x_1$, $f(x) < c$ and when $x > x_2$, $f(x) > c$. Here, we define inverse function $f^{-1}(x)$ which is continue close to 0 at discontinuity, i.e. when $x_2 \leq 0$, $\hat{f}^{-1}(c) = \lim_{x \to c^{+0}} f^{-1}(x) = y_+$;

when $x_1 \geq 0$, $\hat{f}^{-1}(c) = \lim_{x\to c^{-0}} f^{-1}(x) = y_-$; when $0 \in [x_1, x_2]$, define $f^{-1}(c)$ as set of two points $\{y_-, y_+\}$, denote by $\hat{f}^{-1}(c) = \{y_-, y_+\}$.

It is quite straightforward to verify that, if f is an increasing and bounded function on $[-1, 1]$ and $\hat{f}$ is the extensional set-valued function of f, then the inverse function of $\hat{f}$ can be equivalently defined as:

$$\hat{f}^{-1}(x) = \begin{cases} \sup\{t : \hat{f}(t) = x, -1 \leqslant t < 0\}, & -\infty < x \leqslant f(0-) \\ 0, & f(0-) \leqslant x \leqslant f(0+) \\ \inf\{t : \hat{f}(t) = x, 0 < t \leqslant 1\}, & f(0+) \leqslant x < +\infty \end{cases}. \tag{1}$$

Example 2. *To make the above concept more understandable, let consider f as a monotonic bounded function on $[0, 2]$,*

$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant 1 \\ 1 + x, & 1 < x \leqslant 2 \end{cases}.$$

Then its extensional set-valued function $\hat{f}(x)$ and its inverse function $\hat{f}^{-1}(x)$ are defined as the following:

$$\hat{f}(x) = \begin{cases} x, & 0 \leqslant x < 1 \\ [1, 2], & x = 1 \\ 1 + x, & 1 < x \leqslant 2 \end{cases}, \qquad \hat{f}^{-1}(x) = \begin{cases} x, & 0 \leqslant x < 1 \\ 1, & 1 \leqslant x < 2 \\ x - 1, & 2 \leqslant x \leqslant 3 \end{cases}.$$

The $f(x)$ and $\hat{f}(x)$ can be illustrated by Figure 1

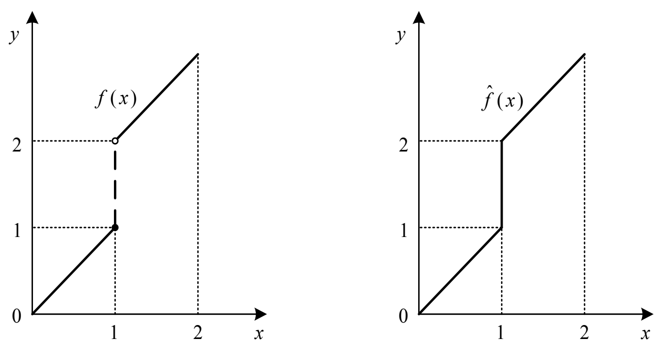

Figure 1: Set-valued function $\hat{f}$ extended by f

4 Fuzzy structured element and transformation

In order to establish the relationship between the fuzzy real number space ε^1 and the monotone function space on interval $[-1, 1]$, we introduce a method, namely the *Fuzzy Structured Element*, which was proposed by Sicong Guo in [14].

Definition 3. Let E be a fuzzy set on real line $\mathbb{R}$ and $E(x)$ is membership function of E. Then, E is called a fuzzy structured element, if $E(x)$ satisfies the following properties:

1) $E(0) = 1$;
2) $E(x)$ is monotonic increasing and right-continuous on $[-1, 0)$, monotonic decreasing and left-continuous on $(0, 1]$;

3) For any $x \in (-\infty, -1) \cup (1, +\infty)$, $E(x) = 0$.

Further, E is called a normal fuzzy structured element if the fuzzy structured element E satisfies: (1) $E(x) > 0$ for all $x \in (-1, 1)$; (2) $E(x)$ is continuous, strictly monotonic increasing on $[-1, 0)$ and also continuous, strictly monotonic decreasing on $(0, 1]$.

According to Definition 3, it is easy to know that the fuzzy structured element is a special fuzzy number on real line $\mathbb{R}$, which can be used to express the concept of fuzzy zero $\tilde{0}$.

Let E be a fuzzy number. E is called a triangular structured element if it has membership function $\mu_E(x)$, where

$$\mu_E(x) = \begin{cases} 1 - x, & x \in [0, 1] \\ 1 + x, & x \in [-1, 0] \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

As it is shown in Figure 2. Obviously, E is a special fuzzy structured element.

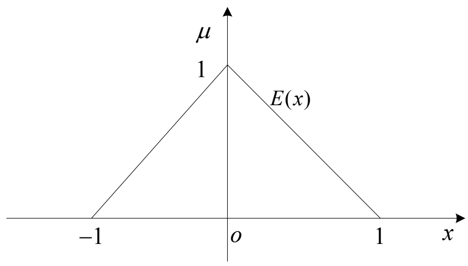

Figure 2: Triangular structured element E

Based on the fuzzy structured element, we can give the following two theorems:

Theorem 4 (Local Mapping Theorem)**.** *Suppose E is a fuzzy structured element on $\mathbb{R}$ with membership function $E(x)$. $f(x)$ is monotonically bounded on $[-1, 1]$ and $\hat{f}(x)$ is extensional set-valued function of $f(x)$. Then $\hat{f}(E)$ is a bounded closed fuzzy number and membership function of $\hat{f}(E)$ is $E(\hat{f}^{-1}(x))$, where $\hat{f}^{-1}(x)$ is the inverse function of $\hat{f}(x)$*
(If $f(x)$ is strictly increasing and continuous on $[-1, 1]$, then $\hat{f}^{-1}(x)$ is a ordinary inverse function of $f(x)$).

Proof: Let $A = \hat{f}(E)$, $A(y)$ defined as a membership function of $\hat{f}(E)$. Suppose that $f(x)$ is increasing and bounded on $[-1, 1]$. By extension principle, we have

$$A(y) = \bigvee_{y \in \hat{f}(x)} E(x), \hat{f}(E) = \bigcup_{x \in \mathbb{R}} E(x) * \hat{f}(x),$$

where

$$E(x) * \hat{f}(x)(y) = \begin{cases} E(x), & y \in \hat{f}(x) \\ 0, & \text{otherwise} \end{cases}.$$

From the former equation, the membership function of $\hat{f}(E)$ is $\hat{f}(E)(y)$. When $y \in \hat{f}(x)$, counterpart membership degree defined as $E(x)$.

Denote α–cut of E by $E_\alpha = [e_\alpha^-, e_\alpha^+]$. It follows from the concept of fuzzy structured element that $E_0 = [e_0^-, e_0^+] \subseteq [-1, 1]$. Since $f(x)$ is increasing bounded on interval $[-1, 1]$ and $\hat{f}(x)$ is surjection on $\mathbb{R}$, it follows that for $\alpha \in (0, 1]$,

$$[\hat{f}(E)]_\alpha = \hat{f}(E_\alpha) = \hat{f}[e_\alpha^-, e_\alpha^+] = [f(e_\alpha^-), f(e_\alpha^+)],$$

For $\alpha = 0$,

$$[\hat{f}(E)]_0 = \overline{\text{supp}\hat{f}(E)} = \overline{\cup_{\alpha\in(0,1]}[\hat{f}(E)]_\alpha} = \overline{\cup_{\alpha\in(0,1]}[\hat{f}(E_\alpha)]} = [f(e_0^-+), f(e_0^+-)] \tag{3}$$

Thus $E_\alpha, \alpha \in [0,1]$ are bounded closed sets.
For all $\alpha_1, \alpha_2 \in (0,1]$, if $\alpha_1 \le \alpha_2$, then $E_{\alpha_1} \subseteq E_{\alpha_2}$, that is $e_{\alpha_2}^- \le e_{\alpha_1}^-, e_{\alpha_1}^+ \le e_{\alpha_2}^+$. Since $f(x)$ is monotone increasing, we have

$$f(e_{\alpha_1}^- -) \le f(e_{\alpha_2}^- -), f(e_{\alpha_2}^+ +) \le f(e_{\alpha_1}^+ +)$$

Therefore, $[\hat{f}(E)]_{\alpha_1} \subseteq [\hat{f}(E)]_{\alpha_2}$. Furthermore, it follows from Eq.(3) that

$$[\hat{f}(E)]_0 = \overline{\cup_{\alpha\in(0,1]}[\hat{f}(E)]_\alpha} \supseteq [\hat{f}(E)]_{\alpha_1},$$

It means that $\hat{f}(E)$ is a convex set on real line $\mathbb{R}$.

Since E is a fuzzy number and 1-cut set E_1 of E is nonempty and $[\hat{f}(E)]_1 = \hat{f}(E_1)$ also is nonempty, hence we can say that $\hat{f}(E)$ is a normal fuzzy number. From definition of bounded closed fuzzy number, we know $\hat{f}(E) \in \tilde{N}_c(R)$.

Since $\hat{f}(E)(y) = \vee_{y\in\hat{f}(x)} E(x) = E(x_y)$ as $y \in \hat{f}(x)$, where $x_y = \hat{f}^{-1}(y)$. It follows that

$$\hat{f}(E)(y) = E(\hat{f}^{-1}(y)),$$

or

$$\hat{f}(E)(x) = E(\hat{f}^{-1}(x)).$$

If $f(x)$ is monotonic decreasing on $[-1,1]$, the proof can be shown in a similar manner. □

Theorem 5 (Theorem of Structured Element Expression of Fuzzy Number). *For a given regular fuzzy structured element E and any bounded fuzzy number A, there exists a monotonic bounded function f on $[-1,1]$ such that $A = f(E)$ (strictly, exists a extended set-valued function $\hat{f}$ such that $A = \hat{f}(E)$). We called it fuzzy number A generated by the fuzzy structured element.*

Proof: From fuzzy number expression theorem, fuzzy number u can be expressed by a family set $\{u_\alpha : u_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)], \alpha \in [0,1]\}$. Therefore, we just need to prove that there exists a monotone bounded function $f(x)$ on $[-1,1]$ such that $f(E) = u$, that is, for all $\alpha \in [0,1]$, $[f(E)]_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)]$.

Let

$$f(x) = \begin{cases} \underline{u}(E(x)), & x \in [-1,0] \\ \overline{u}(E(x)), & x \in (0,1] \end{cases}.$$

It follows from Theorem 1 that $f(x)$ is a monotone increasing bounded function on $[-1,1]$. From the local mapping principle, $f(E)$ is a bounded closed fuzzy number. It follows from the extension principle that $[f(E)]_\alpha = f(E_\alpha)$. Denote $E(x)$ on $[-1,0]$ as $l_E(x)$ and $E(x)$ on $[0,1]$ as $r_E(x)$. Since E is strictly increasing on $[-1,0]$ and is also a bijection from $[-1,0]$ to $[0,1]$, E is strictly decreasing on $[0,1]$ and is also a bijection from $[0,1]$ to $[0,1]$, so $l_E(x), r_E(x)$ inverse and are denoted by $l_E^{-1}(\alpha), r_E^{-1}(\alpha)$, then

$$E_\alpha = [\underline{E}(\alpha), \overline{E}(\alpha)] = [l_E^{-1}(\alpha), r_E^{-1}(\alpha)].$$

Since $\underline{u}(\alpha), \overline{u}(\alpha)$ are left-continuous on $(-1,0]$ and are right-continuous at $\alpha = 0$, also E(x) is continuous, we know that $\underline{u}(E(x)) = \underline{u}(l_E(x))$ is left-continuous on $(-1,0]$ and is right-continuous at $x = -1$, $\overline{u}(E(x)) = \overline{u}(r_E(x))$ is right-continuous on $[0,1)$ and is left-continuous at $x = 1$. Since

$f(x)$ is increasing, it follows that
for all $\alpha \in (0,1]$,

$$\begin{aligned} f(E_\alpha) &= f[l_E^{-1}(\alpha), r_E^{-1}(\alpha)] = [f(l_E^{-1}(\alpha)-), f(r_E^{-1}(\alpha)+)] \\ &= [\underline{u}(E(l_E^{-1}(\alpha)+)), \overline{u}(E(r_E^{-1}(\alpha)-))] \\ &= [\underline{u}(E(l_E^{-1}(\alpha))), \overline{u}(E(r_E^{-1}(\alpha)))] \\ &= [\underline{u}(\alpha), \overline{u}(\alpha)] \end{aligned}$$

for $\alpha = 0$,

$$\begin{aligned} f(E_0) &= f[l_E^{-1}(0), r_E^{-1}(0)] = [f(-1+), f(1-)] \\ &= [\underline{u}(E(-1+)), \overline{u}(E(1-))] \\ &= [\underline{u}(E(-1)), \overline{u}(E(1))] \\ &= [\underline{u}(0), \overline{u}(0)] \end{aligned}$$

Therefore, we conclude that $[f(E)]_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)]$ for all $\alpha \in [0,1]$. That's to say that $f(E) = u$. We complete the proof of this theorem. □

When no confusion can arise in the following discussions, we will use $f(x)$ to denote the extended function $\hat{f}(x)$ and use $f(E)$ to instead of $\hat{f}(E)$, respectively.

Theorem 6. *Let f be a monotonic bounded function and E be a fuzzy structured element on $\mathbb{R}$ and fuzzy number $u = f(E)$. For all $\alpha \in [0,1]$, $E_\alpha = [e_\alpha^-, e_\alpha^+]$. Then*

(1) If $f(x)$ is increasing on $[-1,1]$, then α-cut of fuzzy number u is closed interval

$$u_\alpha = \begin{cases} [f(e_\alpha^- -), f(e_\alpha^+ +)], & \alpha \in (0,1], \\ [f(e_\alpha^- +), f(e_\alpha^+ -)], & \alpha = 0, \end{cases} . \tag{4}$$

(2) If $f(x)$ is monotonic decreasing function on $[-1,1]$, then α-cut of u is closed interval

$$u_\alpha = \begin{cases} [f(e_\alpha^+ -), f(e_\alpha^- +)], & \alpha \in (0,1], \\ [f(e_\alpha^+ +), f(e_\alpha^- -)], & \alpha = 0, \end{cases} \tag{5}$$

Proof: Based on Theorem 4 and function f satisfies the monotone condition of Local Mapping Theorem, it follows that for any $\alpha \in (0,1]$, we have $[f(E)]_\alpha = f(E_\alpha)$. Since f is monotone on closed interval $E_\alpha = [e_\alpha^-, e_\alpha^+] \subseteq [-1,1]$, it follows that:
If f is increasing, for $\alpha \in (0,1]$, we have

$$\begin{aligned} u_\alpha &= f(E_\alpha) = f[e_\alpha^-, e_\alpha^+] \\ &= [\inf\{y : y \in f[e_\alpha^-, e_\alpha^+]\}, \sup\{y : y \in f[e_\alpha^-, e_\alpha^+]\}] \\ &= [f(e_\alpha^- -), f(e_\alpha^+ +)], \end{aligned}$$

and for $\alpha = 0$, it holds that

$$\begin{aligned} u_0 &= \overline{\mathrm{supp} f(E)} = \overline{f(E)_{\dot{0}}} = \overline{f(E_{\dot{0}})} \\ &= \overline{\lim_{\alpha \to 0} [f(e_\alpha^- -), f(e_\alpha^+ +)]} = [f(e_0^- +), f(e_0^+ -)]. \end{aligned}$$

If f is decreasing, we have

$$\begin{aligned} u_\alpha &= f(E_\alpha) = f[e_\alpha^-, e_\alpha^+] \\ &= [\inf\{y : y \in f[e_\alpha^-, e_\alpha^+]\}, \sup\{y : y \in f[e_\alpha^-, e_\alpha^+]\}] \\ &= [f(e_\alpha^+ +), f(e_\alpha^- -)], \end{aligned}$$

□

From the Local Mapping Theorem 4, we know that given a fuzzy structured element E, it will be transformed into an fuzzy number $A = f(E)$ with any a monotonic function f on $[-1, 1]$. When f is not a monotonic function, the fuzzy set $f(A)$ can not be guaranteed to be a fuzzy number. Theorem 5 show us that for any bounded fuzzy number A, we always can find a monotonic bounded function f on $[-1, 1]$ such that $f(E) = A$. Therefore, the two theorems reveal to us that there exists a deep relationship between the family of bounded monotonic function on $[-1, 1]$ and the fuzzy number space.

5 The same order standard monotonic bounded function classes $B[-1, 1]$

Let f be monotonic bounded function on $[-1, 1]$. If for any discontinuity x in $[-1, 1]$, we have

$$f(x) = \frac{1}{2}[f(x+) + f(x-)], \tag{6}$$

where $f(x+)(f(x-))$ is the right-limit(left-limit) of $f(x)$ at the point x, then $f(x)$ is called a standard monotonic bounded function on $[-1, 1]$. All same order standard monotonic bounded function on $[-1, 1]$ is denoted by $B[-1, 1]$.

It is obvious that a continuous monotonic bounded function on $D[-1, 1]$ is also a standard monotonic bounded function.

Definition 7. Suppose that $f \in D[-1, 1]$, we define

$$\check{f}(x) = \begin{cases} f(-1+), & x = -1 \\ [f(x-) + f(x+)]/2, & x \in (-1, 1) \\ f(1-), & x = 1 \end{cases} \tag{7}$$

where $\check{f}(x)$ is as a standardized form of $f(x)$. Obviously, $\check{f} \in B[-1, 1]$. If f is a standard monotonic bounded function, then $\check{f} = f$.

In the following we introduced two distance formulas:

$$d_p(f, g) = \left[\int_{-1}^{1} |f(x) - g(x)|^p \mathrm{d}x\right]^{1/p}, \text{ for all } f, g \in B[-1, 1], \tag{8}$$

$$d_H(f, g) = \sup_{x \in [-1,1]} |f(x) - g(x)|, \text{ for all } f, g \in B[-1, 1]. \tag{9}$$

where $1 \le p < +\infty$.

Theorem 8. *Let E be a normal fuzzy structured element, K is a bounded closed interval on R. Denote*

$$B_f(K) = \{f : f \in B[-1, 1] \text{ and } [f(-1), f(1)] \subseteq K\}, \tag{10}$$

Metric spaces $(B[-1, 1], d_H)$ and $(B_f(K), d_p)$ both are complete.

Proof: First, we prove completeness of space $(B[-1,1],d_p)$. Suppose each elements in $(B[-1,1],d_p)$ are increasing, for decreasing situation has similarity conclusions. Suppose given sequence $\{x_n(t)\}$, where $x_n(t)\in(B[-1,1],d_p), n=1,2,\cdots$.

Let $d_p(x_n,x_m)\to 0,$ (as $n,m\to\infty$), that is, sequence $\{x_n(t)\}$ satisfies Cauchy uniformly convergence conditions. Suppose $x_0(t)$ is limit of sequence $x_n(t)$,i.e.

$$\lim_{n\to\infty} x_n(t)=x_0(t) \text{ for all } t\in[-1,1].$$

Since all $x_n(t_1)$ is monotonic increasing function, then $x_n(t_1)\le x_n(t_2)$ for all $t_1,t_2\in[-1,1]$. So

$$\lim_{n\to\infty} x_n(t_1)\le \lim_{n\to\infty} x_n(t_2),$$

and $x_0(t_1)\le x_0(t_2)$. Thus $x_0(t)$ is monotonic increasing function.

Now we prove $g(t)$ is standard function on $[-1,1]$. Suppose t_0 is a discontinuous point of $g(t)$. We might as well suppose that $g(t)$ is't standard, that is

$$g(t_0)\ne\frac{1}{2}[g(t_0+)+g(t_0-)].$$

Let$\delta=\left|g(t_0)-\frac{1}{2}[g(t_0+)+g(t_0-)]\right|$. Furthermore, since $f_n(t), n=1,2,\cdots$, are standard, we have

$$\begin{aligned}
&E(t_0+)\,|f_n(t_0+)-g(t_0+)|<\varepsilon,\\
&E(t_0-)\,|f_n(t_0-)-g(t_0-)|<\varepsilon,\\
&E(t_0)\,|f_n(t_0)-g(t_0)|<\varepsilon.
\end{aligned}$$

Take $\varepsilon=\varepsilon_0<\delta/2$. When $n\ge n_0(\varepsilon_0)$, we have

$$\begin{aligned}
&|g(t_0)-\frac{1}{2}[g(t_0+)+g(t_0-)]|\\
&\le|f_n(t_0)-g(t_0)|+\frac{1}{2}|f_n(t_0+)-g(t_0+)|+\frac{1}{2}|f_n(t_0-)-g(t_0-)|\\
&\le\varepsilon_0+\varepsilon_0=2\varepsilon_0<\delta.
\end{aligned}$$

a contradiction. Thus, $g(t)$ is standard on $[-1,1]$.

2) Now we prove the metric space $(B_f(K),d_p)$ is complete. Suppose that f_n is a Cauchy sequence in $B_f(K),d_p$, then for any $\varepsilon>0$, there exists a positive integer N such that for any $m,n>N$, we have

$$d_H(f_m,f_n)=\left[\int_{-1}^{1}|f_m(t)-f_n(t)|^p\mathrm{d}x\right]^{1/p}<\varepsilon.$$

This indicates that $\{f_n\}$ is a Cauchy sequence of $L_p[-1,1]$. We know that $L_p[-1,1]$ is complete space, so $\{f_n\}$ is converse in $L_p[-1,1]$. Suppose h is a limit of sequence $\{f_n\}$. Similar to the proof in 1) that $h(x)$ is increasing and bounded in interval $[-1,1]$. Therefore, $h(-1+)$ and $h(1-)$ exist. Let

$$f(x)=\begin{cases} h(-1+), & x=-1\\ \left[h(x+)+h(x-)\right]/2, & x\in(-1,1)\\ h(1-), & x=1\end{cases}.$$

It is obvious that $f(x)\in B[-1,1]$ and also is a limit of Cauchy sequence f_n. $f(x)$ and $h(x)$ have different values which happened only on discontinuity, so we have $d_p(f_n,f)=d_p(f_n,g)$. Thus, f is a limit of Cauchy sequence in $(B_f(K),d_p)$, that is, $\{f_n\}$ converges in $(B_f(K),d_p)$. Hereby, the completeness of $(B_f(K),d_p)$ have been proved. □

Theorem 9. *Let E be a normal fuzzy structured element and $K \in I(\mathbb{R})$ be a nonempty set, denoted by:*

$$B_f(K) = \{f : f \in B[-1,1] \text{ and } [f(-1), f(1)] \subseteq K\},$$

then the metric space $(B_f(K), d_p)$ is complete.

Example 10. *The metric space $(B[-1,1], d_p)$ is not complete. For example, $f_n \in B[-1,1]$ is defined by*

$$f_{n+1}(x) = \begin{cases} f_n(x), & x \in [-1, 1-1/n^2] \\ n, & x \in (1-1/n^2, 1] \end{cases}, (n \geq 1),$$

where $f_1(x) = 0, x \in [-1,1]$. It is obvious that $f_n(x)(n \geq 1)$ are bounded functions. Suppose $m \leq n$, we have

$$\begin{aligned} d_p(f_m, f_n) &= \left[\int_{-1}^{1} |f_m(t) - f_n(t)|^p \mathrm{d}t\right]^{1/p} < \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{m^2}\right]^{1/p} \\ &< \left[\frac{1}{n} - \frac{1}{m}\right]^{1/p} < \frac{1}{n} \to 0(m, n \to \infty) \end{aligned}$$

Thus, $\{f_n\}$ is a Cauchy Sequence, their standard function sequence $\{\check{f}_n(x)\}$ is a Cauchy Sequence in $B[-1,1]$. It is easy to understand that $\{\check{f}_n(x)\}$ do not convergence to any upper bounded function.

Theorem 11. *Metric space $(B[-1,1], d_H)$ is not a separable space and metric space $(B[-1,1], d_p)$ is a separable space.*

Proof: 1) It is sufficient to construct an uncountable set B_f of $B[-1,1]$ with the property that if $f, g \in B_f$, then $d_H(f,g) = 1$. For each $t \in (-1,1)$, define $f_t \in B[-1,1]$ by

$$f_t(x) = \begin{cases} 0 & x \in [-1, t) \\ 0.5 & x = t \\ 1 & x \in (t, 1] \end{cases}.$$

Denote $B_f = \{f_t : t \in (-1,1)\}$. Consequently, if $t_1 \neq t_2$, then $d_H(f_{t_1}, f_{t_2}) = 1$.

2) Since $L_p[-1,1]$ is a separable space with respect to the metric d_p and $B[-1,1] \subset L_p[-1,1]$, then $B[-1,1]$ is separable with respect to metric d_p (For any separable metric space X, any nonempty subset of this space is also separable). □

In general, $f \in B[-1,1], -f \in B[-1,1]$ unless f is a constant-valued function. Because, if f isn't constant-valued function, despite $-f$ is also monotonic function, but it is not same order with f. Hence, $B[-1,1]$ cannot form group with respect to operation of addition,just can form a semigroup.

It should be noted that each element in $B[-1,1]$ is not a closed form with respect to ordinary subtraction operator. We can take an example, function obtain by two monotonic function subtracted may be non-monotonic. Therefore, $B[-1,1]$ can't form linear space with respect to addition and number multiply operation.

Theorem 12. *$B[-1,1]$ is a convex cone with 0 as its vertex.*

The theorem is obvious, so the proof is omitted.

6 Topological relationship between $B[-1,1]$ and $\tilde{N}_c(\mathbb{R})$

6.1 Two types of fuzzy number metric spaces induced by the fuzzy structured element

Let E be a symmetrical regular fuzzy structured element on real line $\mathbb{R}$ and $\tilde{N}_c(\mathbb{R})$ be the set of all bounded closed fuzzy numbers. For given function $f \in B[-1,1]$, there exists corresponding unique fuzzy number such that $A_f = f(E)$. In other words, fuzzy structured element determines a mapping from $B[-1,1]$ to $\tilde{N}_c(\mathbb{R})$.

Denote

$$\begin{aligned} H_E : B[-1,1] &\to \tilde{N}_c(\mathbb{R}), \\ f \quad &\to H_E(f) = f(E) \in \tilde{N}_c(\mathbb{R}). \end{aligned}$$

Then H_E is called fuzzy functional induced by fuzzy structured element E.

Using metrics d_p and d_H on $B[-1,1]$, mapping H_E induces distances

$$d_{Np}(A,B) = d_p(H_E^{-1}(A), H_E^{-1}(B)), \tag{11}$$

$$d_{NH}(A,B) = d_H(H_E^{-1}(A), H_E^{-1}(B)), \tag{12}$$

on $\tilde{N}_c(\mathbb{R})$, where $H_E^{-1}(A), H_E^{-1}(B)$ are preimage of mapping H_E at A and B, respectively. Suppose $A = f(E), B = f(E)$, where $f, g \in B[-1,1]$, then Eq.(11) and Eq.(12) can also rewrite as

$$d_p(f,g) = d_p(H_E(f), H_E(g)), \tag{13}$$

$$d_H(f,g) = d_H(H_E(f), H_E(g)), \tag{14}$$

$(\tilde{N}_c(\mathbb{R}), d_{Np})$ and $(\tilde{N}_c(\mathbb{R}), d_{NH})$ are said to be distance space induced by $(B[-1,1], d_p)$ and $(B[-1,1], d_H)$, respectively. It is easy to understand that H_E is an isometric bijection of $B[-1,1]$ onto $\tilde{N}_c(\mathbb{R})$.

Using isometric bijection H_E, we can translate metric of elements in fuzzy number space to metric between the same order standard monotonic bounded functions in range of $[-1,1]$. Then, what is the relationship between those metrics and the other metrics on fuzzy numbers?

Before discussing the relationship, a Lemma need to be presented here:

For $u \in \tilde{N}_c(\mathbb{R})$, E is a normal fuzzy structured element. If $g \in B[-1,1]$ such that $u = g(E)$, as defined in the following:

$$g_u(x) = \begin{cases} \underline{u}(E(x)) & -1 \le x \le 0 \\ \overline{u}(E(x)) & 0 < x \le 1 \end{cases}.$$

Lemma 13. *Suppose that E is a normal fuzzy structured element, fuzzy number $u \in \tilde{N}_c(\mathbb{R}), u_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)](\alpha \in [0,1])$. If $u = f(E), f \in B[-1,1]$, then*

$$f(x) = \check{g}_u(x), x \in [-1,1],$$

where $\check{g}_u$ is the standard function of g_u.

Proof:

The proof has been provided in the following two steps:

1) First, we prove $\check{g}_u = u$. According to the decomposition theorem, we only need to prove that $[\check{g}_u(E)]_\alpha = u_\alpha$ for any $\alpha \in (0,1]$. Denote $E_\alpha = [e_\alpha^-, e_\alpha^+]$. From the extension principle and $g(x)$ is a increasing function, it follows that

$$[\check{g}_u(E)]_\alpha = \check{g}_u(E_\alpha) = \check{g}_u([e_\alpha^-, e_\alpha^+]) = [\check{g}_u(e_\alpha^- -), \check{g}_u(e_\alpha^+ +)].$$

Since $\underline{u}(\alpha), \overline{u}(\alpha)$ are continuous on $(0,1]$, we have $\underline{u}(E(x))$ is left-continuous on $(-1,0]$ and $\overline{u}(E(x))$ is right-continuous on $[0,1)$. Therefore,

$$\begin{aligned}[\check{g}_u(e_\alpha^- -), \check{g}_u(e_\alpha^+ +)] &= [g_u(e_\alpha^- -), g_u(e_\alpha^+ +)] \\ = \big[\underline{u}\big(E(e_\alpha^-)\big), \overline{u}\big(E(e_\alpha^+)\big)\big] &= \big[\underline{u}(\alpha), \overline{u}(\alpha)\big]\end{aligned}$$

That is, $[\check{g}_u(E)]_\alpha = u_\alpha$. Thus, $\check{g}_u = u$.

2) Here we prove f is unique in $B[-1,1]$. Suppose $f_1, f_2 \in B[-1,1]$ such that $\hat{f}_1(E) = \hat{f}_2(E) = u$, then $\hat{f}_1^{-1} = \hat{f}_2^{-1}$. Furthermore, we have $\hat{f}_1 = \hat{f}_2$, then $f_1 = f_2$. Thus, f is unique in $B[-1,1]$. Therefore, $f(x) = \check{g}_u(x), x \in [-1,1]$. The proof is complete. □

The following theorem shows the relation between the induced fuzzy number metrics d_{NH}, d_{Np} and the previous metrics of fuzzy numbers.

Theorem 14. *Let E be regular structured element, $u, v \in \tilde{N}_c(\mathbb{R})$, there are $f, g \in B[-1,1]$ such that $u = f(E), v = g(E)$. Denote $u_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)], v_\alpha = [\underline{v}(\alpha), \overline{v}(\alpha)]$, then*

$$\begin{aligned} d_{Np}(u,v) &= \left[\int_{-1}^{1} |f(x) - g(x)|^p \mathrm{d}x\right]^{1/p} \\ &= \left[\int_0^1 |\underline{u}(\alpha) - \underline{v}(\alpha)|^p \,\mathrm{d}\underline{E}(\alpha) + |\overline{u}(\alpha) - \overline{v}(\alpha)|^p \,\mathrm{d}\overline{E}(\alpha)\right]^{1/p} \end{aligned} \tag{15}$$

$$d_{NH}(u,v) = \sup_{x\in[-1,1]} |f(x) - g(x)| = \sup_{x\in[-1,1]} \left(|\underline{u}(\alpha) - \underline{v}(\alpha)| \vee |\overline{u}(\alpha) - \overline{v}(\alpha)|\right) \tag{16}$$

Proof: 1) We have from Lemma 13 that

$$d_{Np}(u,v) = d_p(f,g) = d_p(\check{f}_u, \check{g}_v),$$

And $\check{g}_u(x) = g_u(x)$ and $\check{g}_v(x) = g_v(x)$ are bounded almost everywhere on $[-1,1]$ respectively, it follows that $d_p(\check{f}_u, \check{g}_v) = d_p(f_u, g_v)$. Therefore,

$$\begin{aligned} d_{Np}(u,v) = d_p(f_u, g_v) &= \left[\textstyle\int_{-1}^{1} |f_u(x) - g_v(x)|^p \mathrm{d}x\right]^{1/p} \\ = \Big[\textstyle\int_{-1}^{0} |\underline{u}(E(x)) - \underline{v}(E(x))|^p \,\mathrm{d}x &+ \textstyle\int_0^1 |\overline{u}(E(x)) - \overline{v}(E(x))|^p \mathrm{d}x\Big]^{1/p} \end{aligned}$$

Denote $E(x) = l_E(x)$ for $x \in [-1,0]$ and $r_E(x) = E(x)$ for $x \in [0,1]$. Since E is a regular fuzzy structured element, we know that l_E is bijective from $[-1,0]$ to $[0,1]$, r_E is bijective from $[0,1]$ to $[0,1]$. Thus, we can say that l_E^{-1}, r_E^{-1} exist and they are monotone bijections. It obvious that $\underline{E} = l_E^{-1}$ and $\overline{E} = r_E^{-1}$ both are differentiable almost everywhere. Therefore, we have

$$\begin{aligned} d_{Np}(u,v) &= \left[\int_0^1 |\underline{u}(\alpha) - \underline{v}(\alpha)|^p \,\mathrm{d}l_E^{-1}(\alpha) + \int_1^0 |\bar{u}(\alpha) - \bar{v}(\alpha)|^p \mathrm{d}r_E^{-1}(\alpha)\right]^{1/p} \\ &= \left[\int_0^1 |\underline{u}(\alpha) - \underline{v}(\alpha)|^p \,\mathrm{d}\underline{E}(\alpha) - \int_0^1 |\bar{u}(\alpha) - \bar{v}(\alpha)|^p \mathrm{d}\bar{E}(\alpha)\right]^{1/p} \end{aligned}$$

2) From Lemma 13, we obtain

$$d_{NH}(u,v) = d_H(f,g) = d_H\check{f}_u, \check{g}_v),$$

Furthermore, $f_u(x), g_u(x)$ both are left-continuous on $(-1,0]$ and right-continuous on $[0,1)$, and are right-continuous at $x=-1$ and left-continuous at $x=1$, we have

$$\begin{aligned} &|f_u(x) - g_v(x)| \le |f_u(x-) - g_v(x-)| \vee |f_u(x+) - g_v(x+)|, \quad x \in (-1,1) \\ &|f_u(x) - g_v(x)| = |f_u(x) - g_v(x)|, \quad x \in \{-1,1\} \end{aligned}$$

Since $\check{f}_u(x), \check{g}_u(x)$ are standard functions of $f_u(x), g_v(x)$ for $x \in (-1,1)$, we have

$$\begin{aligned} |\check{f}_u(x-) - \check{g}_v(x-)| &= |f_u(x-) - g_v(x-)|; \\ |\check{f}_u(x+) - \check{g}_v(x+)| &= |f_u(x+) - g_v(x+)|. \end{aligned}$$

It follows that $d_H(\check{f}_u, \check{g}_v) = d_H(f_u, g_v)$. Moreover, we have

$$\begin{aligned} d_H(f_u, g_v) &= \sup_{x\in[-1,1]} \{|f_u(x) - g_v(x)|\} \\ &= \sup_{x\in[-1,0]} \{|\underline{u}(E(x)) - \underline{v}(E(x))|\} \vee \sup_{x\in[0,1]} \{|\bar{u}(E(x)) - \bar{v}(E(x))|\} \end{aligned}$$

Let $\alpha = E(x)$ on $[-1,0]$ and $[0,1]$, respectively. Since E is a regular fuzzy structured element, it follows that

$$\begin{aligned} \sup_{x\in[-1,0]} \{|\underline{u}(E(x)) - \underline{v}(E(x))|\} &= \sup_{\alpha\in[0,1]} \{|\underline{u}(\alpha) - \underline{v}(\alpha)|\}, \\ \sup_{x\in[0,1]} \{|\bar{u}(E(x)) - \bar{v}(E(x))|\} &= \sup_{\alpha\in[0,1]} \{|\bar{u}(\alpha) - \bar{v}(\alpha)|\}, \end{aligned}$$

completing the proof of the theorem. □

Since integral variables $\underline{E}(\alpha), \bar{E}(\alpha)$ in the Eq.(6.1) are two general functions, the definite integral is a Riemann-Stieltjes integral. When the E is a triangular structured element (2), we have $\mathrm{d}\underline{E}(\alpha) = \mathrm{d}(\alpha-1) = \mathrm{d}\alpha, \mathrm{d}\bar{E}(\alpha) = \mathrm{d}(-\alpha+1) = -\mathrm{d}\alpha$. Then the Eq.(2) becomes the following form:

$$d_{Np}(u,v) = \left[\int_0^1 |\underline{u}(\alpha) - \underline{v}(\alpha)|^p \,\mathrm{d}\alpha - |\overline{u}(\alpha) - \overline{v}(\alpha)|^p \,\mathrm{d}\alpha\right]^{1/p}.$$

From Eq.(16) in the Theorem 14, we also note that the induced fuzzy number metric $d_{NH}(u,v)$ is the same as the Hausdorff metric(Diamond,1989) [6].

6.2 Homeomorphism between the fuzzy number space $\tilde{N}_c(\mathbb{R})$ and $B[-1,1]$

Proposition 15. *Suppose that $(X, d_X), (Y, d_Y)$ are two metric spaces. F is an isometric bijection from (X, d_X) to (Y, d_Y). Then F is continuous and inverse mapping F^{-1} of F exists and is also continuous.*

Proof: Since F is a bijection of (X, d_X) into (Y, d_Y), there exists inverse mapping F^{-1} which is also one-to-one mapping. By definition of continuous mapping, for all $x_0 \in X$ and any positive number ε, there always exists a positive number δ such that $d_Y(F(x), F(x_0)) < \varepsilon$ as $d_X(x, x_0) < \delta$. Since $d_X(x, x_0) = d_Y(F(x), F(x_0))$, given ε, it is sufficient by taking $\delta \leq \varepsilon$ (For instance, take $\delta = \varepsilon/2$).Hence, F is continuous. Similarly, we can also prove inverse mapping F^{-1} which is also continuous. □

Given a bounded closed interval K, let the uniformly bounded fuzzy number set as

$$\tilde{N}_c(K) = \{u : u \in \tilde{N}_c(\mathbb{R}) \text{ and supp } u \subseteq K\},$$

Since there exists a bijection H_E of $B[-1,1]$ into $\tilde{N}_c(\mathbb{R})$ and H_E and inverse function H_E^{-1} are continuous, thus we have conclusions as follows:

Theorem 16. *Metric spaces $(B[-1,1], d_H)$ and $(\tilde{N}_c(\mathbb{R}), d_{NH})$ are homeomorphic. Metric spaces $(B_f(K), d_p)$ and $(\tilde{N}_c(K)), d_{Np})$ are homeomorphic.*

Since space $(\tilde{N}_c(\mathbb{R}), d_{NH})$ and $(B[-1,1], d_H)$ are homeomorphic, that is, both metric spaces are topologically equivalent. So elements in both of them have consistent properties on metrics. There are one-to-one relationship between fuzzy number sequence $\{u_n\}$ of $(\tilde{N}_c(\mathbb{R}), d_{NH})$ and function sequence $\{f_n\}$, fuzzy number sequence on $(\tilde{N}_c(\mathbb{R}), d_{NH})$ and function sequence on $(B[-1,1], d_H)$ have completely same properties. Similarly, fuzzy number sequence on $(\tilde{N}_c(K), d_{Np})$ and function sequence on $(B_f(K), d_H)$ have completely same properties. Therefore, the properties of convergence sequence of general metric spaces are also founded to the convergence fuzzy number sequence. Thus, they are trivial to the following corollaries.

Corollary 17. *Fuzzy number metric space $(\tilde{N}_c(\mathbb{R}), d_{NH})$ is complete and $(\tilde{N}_c(\mathbb{R}), d_{Np})$ is not complete.*

Corollary 18. *For any nonempty closed interval K on $\mathbb{R}$, $(\tilde{N}_c(K), d_{Np})$ is a complete metric space.*

Corollary 19. *For any fuzzy number sequence $\{u_n\}$ of fuzzy number space $(\tilde{N}_c(\mathbb{R}), d_N)$ only has a limit almost everywhere. That is, the limit of convergence sequence is unique.(Where d_N represents d_{NH} or d_{Np}).*

Corollary 20. *Suppose that $\{u_n\}$ is a fuzzy number sequence of $(\tilde{N}_c(\mathbb{R}), d_{NH})$, $\{f_n\}$ is a function sequence on $(B[-1,1], d_H)$. For all $n \geq 1$, $u_n = f_n(E)$. Then fuzzy number sequence $\{u_n\}$ converge if and only if function sequence $\{f_n\}$ converge. Let*

$$\lim_{n\to\infty} u_n = u_0, \lim_{n\to\infty} f_n = f_0,$$

then $u_0 = f_0(E)$.

If using d_{Np} , d_p, $B_f(K)$ (defined as 10) instead of d_{NH}, d_H and $B[-1,1]$ in Corollary 20, respectively. The Corollary is still founded.

7 Conclusion

By using monotonic mapping of the fuzzy structured element, we have proved that the bounded fuzzy number space is homeomorphic to the space $B[-1,1]$ of monotonic bounded function with same monotonicity on $[-1,1]$. Therefore, the problem of the fuzzy number space can be transformed to one's of space $B[-1,1]$, such as the convergence of sequence of fuzzy numbers, the convergence of the fuzzy series , continuous of fuzzy-valued function and so on. To some extent, our study provides a new way for the study of fuzzy analysis.

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Fuzzy Membrane Computing: Theory and Applications

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Abstract: Fuzzy membrane computing is a newly developed and promising research direction in the area of membrane computing that aims at exploring the complex interaction between membrane computing and fuzzy theory. This paper provides a comprehensive survey of theoretical developments and various applications of fuzzy membrane computing, and sketches future research lines. The theoretical developments are reviewed from the aspects of uncertainty processing in P systems, fuzzification of P systems and fuzzy knowledge representation and reasoning. The applications of fuzzy membrane computing are mainly focused on fuzzy knowledge representation and fault diagnosis. An overview of different types of fuzzy P systems, differences between spiking neural P systems and fuzzy reasoning spiking neural P systems and newly obtained results on these P systems are presented.
Keywords: fuzzy membrane computing, fuzzy set, multi-fuzzy set, membrane computing, fuzzy reasoning spiking neural P systems, trapezoidal fuzzy number, linguistic term.

1 Introduction

Membrane computing (MC), introduced by Gh. Păun in [1], is an attractive research field of computer science aiming at abstracting computing models, called membrane systems or P systems, from the structures and functioning of living cells, as well as from the way the cells are organized in tissues or higher order structures. Since then, the MC research has been continuously and rapidly progressing [2], [3]. There are, basically, three main types of P systems: cell-like P systems, tissue-like P systems and neural-like P systems [3]. For all cases, a P system consists of three basic elements: membrane structure, multisets of objects and evolution rules. In recent years, the research on neural-like P systems mainly focused on spiking neural P systems (SN P systems), which were introduced in [4] being a type of P systems inspired by the neurophysiological behavior of neurons sending electrical impulses (spikes) along axons from presynaptic neurons to postsynaptic neurons in a distributed and parallel manner. Recently, SN P systems have become a hot topic in membrane computing [2, 5]- [25].

A fuzzy set is a class of objects with a continuum of grades of membership and is characterized by a membership (characteristic) function which assigns to each objects a grade of membership ranging in [0, 1]. The theory of fuzzy sets was proposed by Zadeh in [26] as an extension of the classical notion of set. Since then fuzzy set theory has been studied extensively over the past several decades years. Most of the early interests in fuzzy set theory pertained to representing uncertainty in human cognitive processes [27]. Fuzzy set theory is now combined with other methods and applied to problems in engineering, business, medical and related health sciences, the natural sciences and so on [27]- [34].

P systems are models of computation inspired by the structure and functioning of cells in living organisms. Since uncertainty is an inherent property of all living systems, an increasing interest in the development of uncertain mathematical approaches to membrane computing is emerging due to three main reasons: keeping characteristics of P systems closer to the non-crisp behavior of real cells; the development of new formal computational models dealing with fuzzy information; the possibility of applying P systems to model real biological processes, where handling uncertainty, errors and approximations is necessary. Păun mentioned this in his first list of open problems in membrane computing for the development of approximate mathematical approaches in [35] and this topic continued to be discussed in [36]- [38].

A first contribution to this research line was given in [39], by extending the classical model to several probabilistic ones. Several possible rough set based mathematical models of uncertainty that could be used in membrane computing were further discussed in [40], [41]. Meanwhile, several fuzzy approaches based on fuzzy set theory have been introduced [12]- [25] [42]- [48]. The basic idea of fuzzifying P systems is the substitution of one or all ingredients in a P system with their fuzzy counterparts to obtain fuzzy P systems. Fuzzy P systems possess the same structure as crisp P systems but admit the association of numerical fuzzy values to elements in the membranes. Up to now, several aspects from fuzzy theory has been introduced into cell-like P systems [42]- [48] and SN P systems [12]- [25]. Fuzzy cell-like P systems mainly focused on theoretical research while fuzzy reasoning spiking neural P systems (FRSN P systems) concentrated on both theoretical and application researches. To date, five types of FRSN P systems were proposed for knowledge representation and reasoning as well as used for fault diagnosis. These FRSN P systems are fuzzy reasoning spiking neural P systems with real numbers (rFRSN P systems), fuzzy reasoning spiking neural P systems with linguistic terms (lFRSN P systems), adaptive fuzzy reasoning spiking neural P systems with real numbers (AFRSN P systems), weighted fuzzy reasoning spiking neural P systems (WFRSN P systems) and fuzzy reasoning spiking neural P systems with trapezoidal fuzzy numbers (tFRSN P systems).

Fuzzy membrane computing (FMC) has well established theoretical foundation and real world applications. To outline the work in the past twelve years on fuzzy membrane computing, this paper reviews and summarizes theoretical developments and various applications of fuzzy membrane computing. This comprehensive survey provides an overview of newly developed and promising research lines in the area of membrane computing as well as lists some promising research topics. As such, this work will be of interest to advance in the research line of fuzzy membrane computing, as well as to members of the membrane computing and fuzzy theory communities, specially newcomers.

The remainder of this paper is organized as follows. Section 2 provides an overview of theoretical achievements of FMC. The overview of the application development is outlined in Section 3. Finally, some conclusions and possible further developments are discussed.

2 Theoretical development

This section will present theoretical developments of FMC on the aspects of handling uncertainty in P systems, fuzzification of P systems and fuzzy knowledge representation and reasoning.

2.1 Uncertainty processing in P systems

MC is a formal computational model that rewrites multisets of objects within a spatial structure inspired by membrane structures and functioning of living cells, as well as from the way the cells are organized in tissues or higher order structures and expresses biochemical processes taking place inside cells by evolution rules. Features of "exact" membrane computing developed so far are summarized as follows: objects used in computations are exact copies of reactants involved in biochemical reactions modeled by the rules; an application of a given rule always yields exact copies of objects that the rule is assumed to produce. However, in real life, cells do not behave only in this "exact" way. Biochemical reactions may deal with inexact, uncertain and mutated copies of reactants involved in the cells, and errors may happen accompanying with a biochemical reaction. This means that since inaccuracy and uncertainty exist in real cells i.e. real cells show a non-crisp behavior, inaccuracy and uncertainty should be considered when MC models are established. In this way, actual objects produced when applying rules during computations, and the actual objects conforming the result of computations themselves, would not need to be exact copies of the reactants that are assumed to be used when applying the rules along the computations, but only approximate copies of these reactants.

In [42]- [44], a fuzzy symport/antiport membrane system that uses inexact copies of reactants in the transitions was proposed to handle this kind of inaccuracy and uncertainty. Then, universality of this fuzzy model of computation was proved. The key ingredients in this model are the use of fuzzy multisets in configurations, the endowment of evolution rules with threshold mappings that determine the degree of accuracy of objects to reactants in order to be affected by the rules, and an appropriate way of evaluating the content of the output membrane at the end of a halting computation. This is a first step towards the use of fuzzy methods to answer a question proposed by Păun in the last problem of his first list of open problems in membrane computing [35]. In [44], a fuzzy model of P systems in which the objects involved in computations were colored by means of a finite family of fuzzy sets, and several applications of this model in computational biology were discussed. In [42]- [44], fuzzy methods were used to cope with the possibility that the objects in the membranes were imperfect, approximate copies of the reactants involved in the reactions.

Moreover, in [45], a t-norm based approach for handling imprecision was proposed in P systems. In this work, a P system with vague boundaries in the t-norm approach was proposed and only string-objects were considered inside the membrane structure.

2.2 Fuzzification of P systems

Uncertainty is an inherent property of all living systems. Curiously enough, computational models inspired by biological systems do not take, in general, under consideration this essential aspect of living systems. P systems are computing models inspired by the structure and functioning of cells in living organisms. Thus, it is more than necessary to introduce uncertainty into P system models to fuzzify them due to the development of new formal computational paradigms dealing with fuzzy information and the possibility of applying P systems to model real biological processes, where handling uncertainty, errors and approximations is necessary.

In [46], fuzzy set theory and fuzzy logic were considered in the framework of P systems in order to deal with imprecise biological information. The motivation of this research was to use P systems for modeling the functioning of specific cellular structures and phenomena and, as a final goal, making P systems useful and relevant tools for biologists, and hence motivating further cooperation among scientists working in the areas of P systems and microbiology.

In [47], two variants of P systems with fuzzy components, P systems with fuzzy data and P systems with fuzzy multiset rewriting rules, were introduced. The basic idea behind this attempt to fuzzify P systems was the substitution of one or all ingredients of a P system with their fuzzy counterparts. In this work, the theory of multi-fuzzy sets was developed and the notion of a fuzzy multiset rewriting rule was presented to define P systems with the aforementioned fuzzy components, fuzzy data and fuzzy multiset rewriting rules. By silently assuming that fuzzy data were not the result of some fuzzification process, P systems with fuzzy data were shown to be a step towards real hypercomputation, while P systems with fuzzy multiset rewriting rules were shown to be equivalent to fuzzy Turing machines. Besides, the idea of P systems with both fuzzy data and fuzzy multiset rewriting rules was briefly discussed.

In [41], general fuzzy membrane systems with general multi-fuzzy sets and their evolution rules were introduced by using general multi-fuzzy sets to fuzzification of multisets and P systems. This kind of fuzzy membrane systems was motivated by some practical applications in biochemistry and medical science, where weakness causing uncertainty of an occurrence of an object in a system was determined not only by the number of occurrence copies of that object, but also by a quality of these copies.

In [48], an orthogonal approach for the fuzzification of both multisets and hybrid sets was presented. In this work, L-multi-fuzzy and L-fuzzy hybrid sets were introduced to P systems and a general fuzzy P system with L-multi-fuzzy sets was proposed. Although replacing the multi-fuzzy sets from [41] with L-multi-fuzzy sets did not improve the computational power of the resulted P systems, P systems with L-multi-fuzzy sets might be quite useful in modeling living organisms.

2.3 Fuzzy knowledge representation

Fuzzy reasoning spiking neural P systems (FRSN P systems) are proposed to handle fuzzy knowledge [12]- [25]. Until now, five classes of FRSN P systems, rFRSN P systems, lFRSN P systems, AFRSN P systems, WFRSN P systems and tFRSN P systems, have been proposed to represent fuzzy knowledge. In what follows, we first summarize fuzzy production rules used for fuzzy knowledge representation. Subsequently, we survey these FRSN P systems with respect to their development and definitions, providing for each of them the corresponding models for the considered fuzzy production rules.

Fuzzy production rules

Fuzzy production rules are widely used in fuzzy knowledge representation [29]- [30]. Fuzzy production rules consist of five types: simple fuzzy production rules, composite fuzzy conjunctive rules in the antecedent, composite fuzzy conjunctive rules in the consequent, composite fuzzy disjunctive rules in the antecedent and composite fuzzy disjunctive rules in the consequent. Rules are discussed below:

A *simple fuzzy production rule* is of the form

$$\text{Type 1 } R_i\text{: IF } p_j \text{ THEN } p_k \text{ (CF=}\tau_i) \tag{1}$$

where R_i indicates the ith fuzzy production rule; $\tau_i \in [0,1]$ represents its certainty factor; p_j and p_k represents two propositions, each of which has a fuzzy truth value in $[0,1]$. If fuzzy truth values of propositions p_j and p_k are α_j and α_k, respectively, then $\alpha_k = \alpha_j * \tau_i$.

A *composite fuzzy conjunctive rule in the antecedent* is of the form

$$\text{Type 2 } R_i\text{: IF } p_1 \text{ and } \cdots \text{ and } p_{k-1} \text{ THEN } p_k \text{ (CF=}\tau_i) \tag{2}$$

where $k \geq 3$, R_i indicates the ith fuzzy production rule; $\tau_i \in [0,1]$ represents its certainty factor; $p_1, \cdots, p_{k-1}$ are propositions in the antecedent part of the rule; p_k is a proposition in the consequent part of the rule. If fuzzy truth values of propositions $p_1, \cdots, p_k$ are $\alpha_1, \cdots, \alpha_k$, respectively, then $\alpha_k = \min(\alpha_1, \ldots, \alpha_{k-1}) * \tau_i$.

A *composite fuzzy conjunctive rule in the consequent* is of the form

$$\text{Type 3 } R_i\text{: IF } p_1 \text{ THEN } p_2 \text{ and } \cdots \text{ and } p_k \quad (\text{CF}=\tau_i) \tag{3}$$

where $k \geq 3$, R_i indicates the ith fuzzy production rule; $\tau_i \in [0,1]$ represents its certainty factor; p_1 is a proposition in the antecedent part of the rule; $p_2, \cdots, p_k$ are propositions in the consequent part of the rule. If fuzzy truth values of proposition $p_1, \cdots, p_k$ is $\alpha_1, \cdots, \alpha_k$, then $\alpha_2 = \alpha_1 * \tau_i, \ldots, \alpha_k = \alpha_1 * \tau_i$.

A *composite fuzzy disjunctive rules in the antecedent* is of the form

$$\text{Type 4 } R_i\text{: IF } p_1 \text{ or } \cdots \text{ or } p_{k-1} \text{ THEN } p_k \ (\text{CF}=\tau_i) \tag{4}$$

where $k \geq 3$, R_i indicates the ith fuzzy production rule; $\tau_i \in [0,1]$ represents its certainty factor; $p_1, \cdots, p_{k-1}$ are propositions in the antecedent part of the rule; p_k is a proposition in the consequent part of the rule. If fuzzy truth values of propositions $p_1, \cdots, p_k$ are $\alpha_1, \cdots, \alpha_k$, respectively, then $\alpha_k = \max(\alpha_1, \ldots, \alpha_{k-1}) * \tau_i$.

A *composite fuzzy disjunctive rule in the consequent* is of the form

$$\text{Type 5 } R_i\text{: IF } p_1 \text{ THEN } p_2 \text{ or } \cdots \text{ or } p_k \ (\text{CF}=\tau_i) \tag{5}$$

where R_i indicates the ith fuzzy production rule; $\tau_i \in [0,1]$ represents its certainty factor; p_1 is a proposition in the antecedent part of the rule; $p_2, \cdots, p_k$ are propositions in the consequent part of the rule. This type of rules is unsuitable for knowledge representation due to the fact that it does not make any specific implication [29]. Thus, this type of rules is not discussed here and will not be considered in the following sections.

rFRSN P systems

Fuzzy reasoning spiking neural P systems with real numbers (rFRSN P systems) were first introduced in [12] and further investigated in [13]- [15]. The definition of an rFRSN P system is described in Definition 1.

Definition 1. An rFRSN P system of degree $m \geq 1$ is a tuple $\Pi = (A, \sigma_1, \ldots, \sigma_m, syn, I, O)$, where:

1. $A=\{a\}$ is the singleton alphabet (the object a is called spike);
2. $\sigma_1, \ldots, \sigma_m$ are neurons, of the form $\sigma_i=(\alpha_i, \tau_i, r_i)$ with $i \in \{1, \ldots, m\}$, where
   (a) $\alpha_i \in [0,1]$ is a real number representing the (potential) value of spike contained in neuron σ_i (also called pulse value);
   (b) $\tau_i \in [0,1]$ is a real number representing the truth value associated with neuron σ_i;
   (c) r_i is a firing/spiking rule contained in neuron σ_i, of the form $E/a^\alpha \to a^\beta$, where α,$\beta \in [0,1]$, $E = a^n$ is the firing condition and n represents the number of input synapses from other neurons to this neuron. The firing condition $E = a^n$ indicates that if the neuron receives n spikes, the spiking rule can be applied; otherwise the rule cannot be enabled until n spikes are received. When the number of spikes received by a neuron is less than n, value of the spikes received will be updated according to logical AND or OR operations. For neuron σ_i, if its firing rule $E/a^\alpha \to a^\beta$ can be applied, then the neuron fires. This means its pulse value $\alpha > 0$ is consumed (removed) and it produces a spike with value β;

3 $syn \subseteq \{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\}$ with $i \neq j$ for all $(i, j) \in syn$, $1 \leq i, j \leq m$ (*synapses* between neurons);

4 I and O are nonempty sets that represent input neuron set and output neuron set, respectively.

In order to use fuzzy production rules for fuzzy knowledge representation, we need map them into rFRSN P systems. In the following, we summarize rFRSN P systems for fuzzy production rules, where value τ_i of each rule neuron is assigned to the certainty factor of the fuzzy production rule associated with it.

A *simple fuzzy production rule* can be modeled by an rFRSN P system Π_1, as shown in Figure 1(a), being $\Pi_1 = (A, \sigma_i, \sigma_j, \sigma_k, syn, I, O)$, where

(1) $A = \{a\}$

(2) σ_i is a rule neuron associated with rule R_i with confidence factor τ_i. Its spiking rule is of the form $E/a^{\alpha} \to a^{\beta}$, where $\beta = \alpha * \tau_i$.

(3) σ_j and σ_k are two proposition neurons associated with propositions p_j and p_k with truth values α_j and α_k, respectively. Their spiking rules are of the form $E/a^{\alpha} \to a^{\alpha}$.

(4) $syn = \{(j, i), (i, k)\}$, $I = \{\sigma_j\}$, $O = \{\sigma_k\}$.

A *composite fuzzy conjunctive rule in the antecedent* can be modeled by an rFRSN P system Π_2, as shown in Figure 1(b), being $\Pi_2 = (A, \sigma_1, \sigma_2, \ldots, \sigma_k, \sigma_{k+1}, syn, I, O)$, where

(1) $A = \{a\}$

(2) $\sigma_j (j = 1, 2, \ldots, k)$ are proposition neurons associated with propositions $p_j (j = 1, 2, \ldots, k)$ with truth values $\alpha_j (j = 1, 2, \ldots, k)$, respectively. Their spiking rules are of the form $E/a^{\alpha} \to a^{\alpha}$.

(3) σ_{k+1} is an "AND"-type rule neuron associated with rule R_i with confidence factor τ_i. Its spiking rule is of the form $E/a^{\alpha} \to a^{\beta}$, where $\beta = \alpha * \tau_i$.

(4) $syn = \{(1, k+1), (2, k+1), \ldots, (k-1, k+1), (k+1, k)\}$.

(5) $I = \{\sigma_1, \sigma_2, \ldots, \sigma_{k-1}\}$, $O = \{\sigma_k\}$.

A *composite fuzzy conjunctive rule in the consequent* can be modeled by an rFRSN P system Π_3, as shown in Figure 1(c), being $\Pi_3 = (A, \sigma_1, \sigma_2, \ldots, \sigma_k, \sigma_{k+1}, syn, I, O)$, where

(1) $A = \{a\}$

(2) $\sigma_j (j = 1, 2, \ldots, k)$ are proposition neurons associated with propositions $p_j (j = 1, 2, \ldots, k)$ with truth values $\alpha_j (j = 1, 2, \ldots, k)$, respectively. Their spiking rules are of the form $E/a^{\alpha} \to a^{\alpha}$.

(3) σ_{k+1} is a rule neuron associated with rule R_i with confidence factor τ_i. Its spiking rule is of the form $E/a^{\alpha} \to a^{\beta}$, where $\beta = \alpha * \tau_i$.

(4) $syn = \{(1, k+1), (k+1, 2), (k+1, 3), \ldots, (k+1, k)\}$.

(5) $I = \{\sigma_1\}$, $O = \{\sigma_2, \sigma_3, \ldots, \sigma_k\}$.

A *composite fuzzy disjunctive rule in the antecedent* can be modeled by an rFRSN P system Π_4, as shown in Figure 1(d), being $\Pi_4 = (A, \sigma_1, \sigma_2, \ldots, \sigma_k, \sigma_{k+1}, syn, I, O)$, where

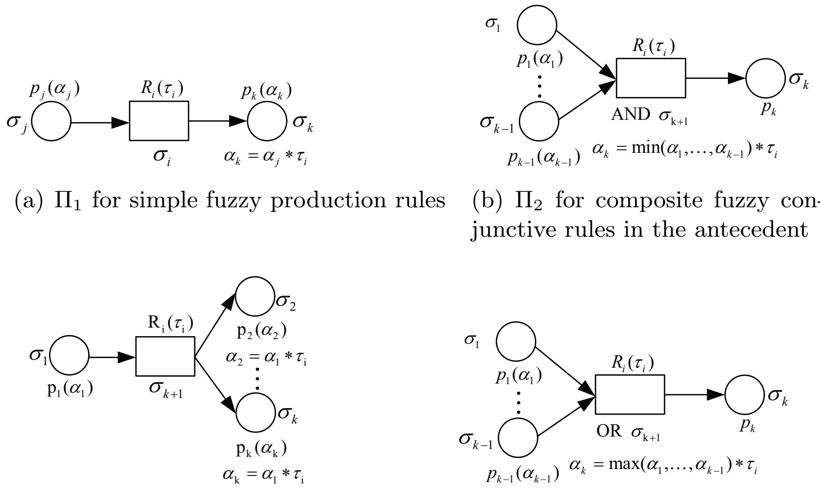
(1) $A = \{a\}$

(2) $\sigma_j (j = 1, 2, \ldots, k)$ are proposition neurons associated with propositions $p_j (j = 1, 2, \ldots, k)$ with truth values $\alpha_j (j = 1, 2, \ldots, k)$ respectively. Their spiking rules are of the form $E/a^{\alpha} \to a^{\alpha}$.

(3) σ_{k+1} is an "OR"-type rule neuron associated with rule R_i with confidence factor τ_i. Its spiking rule is of the form $E/a^{\alpha} \to a^{\beta}$, where $\beta = \alpha * \tau_i$.

(4) $syn = \{(1, k+1), (2, k+1), \ldots, (k-1, k+1), (k+1, k)\}$.

(5) $I = \{\sigma_1, \sigma_2, \ldots, \sigma_{k-1}\}$, $O = \{\sigma_k\}$.



(a) Π_1 for simple fuzzy production rules

(b) Π_2 for composite fuzzy conjunctive rules in the antecedent

(c) Π_3 for composite fuzzy conjunctive rules in the consequent

(d) Π_4 for composite fuzzy disjunctive rules in the antecedent

Figure 1: rFRSN P systems for fuzzy production rules

lFRSN P systems

The first version of fuzzy reasoning spiking neural P systems with linguistic terms (lFRSN P systems) was introduced in [16] and further expanded in [17]. The definition of a lFRSN P system is described in Definition 2.

Definition 2. An lFRSN P system of degree $m \geq 1$ is a tuple $\Pi = (O, P, R, Q, syn, in, out)$, where:

1. O=$\{a\}$ is the singleton alphabet (the object a is called spike);

2. $P = \{p_1, \ldots, p_k\}$ is a finite set of fuzzy propositions, where p_i represents the ith fuzzy proposition, $1 \leq i \leq k$;

3. $R = \{R_1, \ldots, R_n\}$ is a finite set of fuzzy productions rules, where R_i represents the ith fuzzy production rule, $1 \leq i \leq n$;

4. $Q = \{\sigma_1, \dots, \sigma_m\}$ is a finite set of neurons, where $Q = Q_1 \cup Q_2$, $Q_1 \cap Q_2 = \emptyset$. $Q_1 = \{\sigma_1, \dots, \sigma_k\}$ is a set of fuzzy proposition neurons and each proposition neuron i in Q_1 corresponds to a fuzzy proposition p_i in P, where $1 \le i \le k$. $Q_2 = \{\sigma_{k+1}, \dots, \sigma_{k+n}\}$ is a set of rule neurons and each rule neuron $k+i$ in Q_2 corresponds to a fuzzy production rule R_i in R, where $1 \le i \le n$, $m = k + n$. Each neuron in Q has the form of $\sigma_i = (A_i, C_i, r_i)$, $1 \le i \le m$, where:
   (a) A_i is a linguistic term representing potential value contained in σ_i;
   (b) For proposition neurons in Q_1, C_i is ignored; for rule neurons in Q_2, C_i is a linguistic term representing the certainty factor of the corresponding fuzzy production rule of ith neuron;
   (c) r_i is a firing/spiking rule contained in neuron σ_i, of the form $a^\mu \to a^\nu$, where μ and ν are linguistic terms representing potential values of neurons. For rule neurons, μ is logical "and" (denoted by $\otimes$) or "or" (denoted by $\oplus$) of all inputs received by this neuron, and $\nu = \mu \otimes C_i$;
5. $syn \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, m\}$ with $i \neq j$ for all $(i, j) \in syn$, $1 \le i, j \le m$ (*synapses* between neurons);
6. *in* and *out* are input neuron set and output neuron set, respectively.

In what follows, we summarize lFRSN P system models for rules *Type 1* to *Type 4*, as shown in Figure 2.

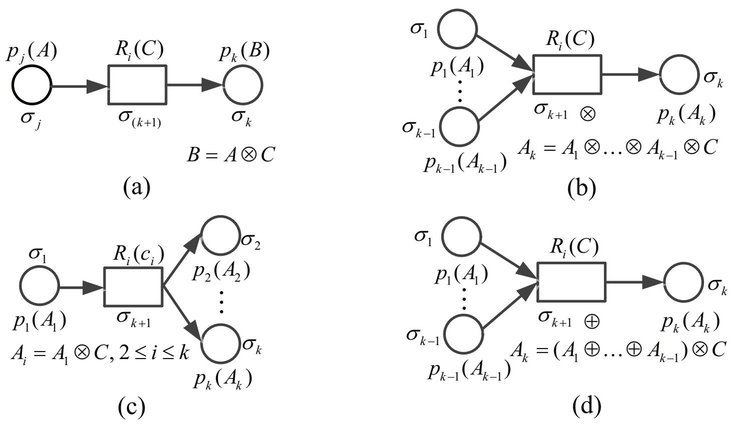


Figure 2: lFRSN P system models for fuzzy production rules. (a) *Type 1*; (b) *Type 2*; (c) *Type 3*; (d) *Type 4*.

Type 1 R_i: IF p_j THEN p_k (CF=C). The fuzzy truth value of proposition p_k is $B = A \otimes C$, where $A_\alpha = [a_1^\alpha, a_2^\alpha]$, $C_\alpha = [c_1^\alpha, c_2^\alpha]$, $\alpha \in [0, 1]$, i.e., $B = \int_0^1 \alpha[a_1^\alpha \wedge c_1^\alpha, a_2^\alpha \wedge c_2^\alpha]$.

Type 2 R_i: IF p_1 and $\cdots$ and p_{k-1} THEN p_k (CF=C). The fuzzy truth value of proposition p_k is $A_k = A_1 \otimes \dots \otimes A_{k-1} \otimes C$, where $[a_{i1}^\alpha, a_{i2}^\alpha]$ is α-cut of A_i, $1 \le i \le k-1$, $\alpha \in [0, 1]$, i.e., $A_k = \int_0^1 \alpha[a_{11}^\alpha \wedge a_{(k-1)1}^\alpha \wedge \dots \wedge c_1^\alpha, a_{12}^\alpha \wedge \dots \wedge a_{(k-1)2}^\alpha \wedge c_2^\alpha]$.

Type 3 R_i: IF p_1 THEN p_3 and $\cdots$ and p_k (CF=C). The fuzzy truth value of propositions $p_2, \dots, p_k$ are identical, i.e., $A_i = A_1 \otimes C$, $2 \le i \le k$, where $[a_1^\alpha, a_2^\alpha]$ is α-cut of A_1, $\alpha \in [0, 1]$, i.e., $A_i = \int_0^1 \alpha[a_1^\alpha \wedge c_1^\alpha, a_2^\alpha \wedge c_2^\alpha]$, $2 \le i \le k$.

Type 4 R_i: IF p_1 or p_2 or $\cdots$ or p_{k-1} THEN p_k (CF=C). The fuzzy truth value of proposition p_k is $A_k = (A_1 \otimes \ldots \oplus A_{k-1}) \otimes C$, where $[a_{i1}^{\alpha}, a_{i2}^{\alpha}]$ is α-cut of A_i, $1 \leq i \leq k-1$, $\alpha \in [0,1]$, i.e., $A_k = \int_0^1 \alpha[(a_{11}^{\alpha} \vee \ldots \vee a_{(k-1)1}^{\alpha}) \wedge c_1^{\alpha}, (a_{12}^{\alpha} \vee \ldots \vee a_{(k-1)2}^{\alpha}) \wedge c_2^{\alpha}]$.

AFRSN P systems

Adaptive fuzzy reasoning spiking neural P systems (AFRSN P systems) were first introduced in [20] and further investigated in [21]. The definition of an AFRSN P system is described in Definition 3 [21].

Definition 3. An AFRSN P system (*of degree* $m \geq 1$) is a tuple $\Pi = (A, N_p, N_r, syn, I, O)$, where

- A=$\{a\}$ is the singleton alphabet (the object a is called spike);
- $N_p = \{\sigma_{p1}, \ldots, \sigma_{pm}\}$ is proposition neuron set, where proposition neuron σ_{pi} expresses the ith proposition of weighted fuzzy production rules, $1 \leq i \leq m$. $\sigma_{pi} = (\alpha_i, \omega_i, \lambda_i, r_i)$, where
  - $\alpha_i \in [0,1]$ is the pulse value contained in proposition neuron σ_{pi}. α_i is used to express fuzzy truth value of the proposition associated with proposition neuron σ_{pi};
  - $\omega_i = (\omega_{i1}, \ldots, \omega_{is_i})$ expresses the output weight vector of neuron σ_{pi}, where $\omega_{ij} \in [0,1]$ is the weight on jth output synapse of the neuron, $1 \leq j \leq s_i$, and s_i is the number of all output synapses of the neuron;
  - r_i is a firing/spiking rule, of the form $E/a^{\alpha} \to a^{\alpha}$, $\alpha \in [0,1]$. $E = \{\alpha \geq \lambda_i\}$ is the firing condition, i.e., if $\alpha \geq \lambda_i$, then the firing rule will be enabled, where $\lambda_i \in [0,1)$ is called the firing threshold;
- $N_r = \{\sigma_{r1}, \ldots, \sigma_{rn}\}$ is rule neuron set, where rule neuron σ_{ri} expresses the ith weighted fuzzy production rule, $1 \leq i \leq n$. $\sigma_{ri} = (\alpha_i, \gamma_i, \tau_i, r_i)$, where
  - $\alpha_i \in [0,1]$ is the potential value (spike) contained in rule neuron σ_{ri};
  - $\gamma_i \in [0,1]$ is the certain factor, which represents the strength of belief of the weighted fuzzy production rule associated with rule neuron σ_{ri}. α_i is also the weight on output synapse (arc) of the neuron;
  - r_i is a firing/spiking rule, of the form $E/a^{\alpha} \to a^{\beta}$, where $\alpha, \beta \in [0,1]$. $E = \{\alpha \geq \tau_i\}$ is the firing condition, i.e., if $\alpha \geq \tau_i$, then the firing rule will be enabled, where $\tau_i \in [0,1)$ is called the firing threshold;
- $syn \subseteq (N_p \times N_r) \bigcup (N_r \times N_p)$ indicates synapses between proposition neurons and rule neurons. Note that there are no synapse connections between any two proposition neurons or between any two rule neurons;
- $I, O \subseteq N_p$ are input neuron set and output neuron set, respectively.

The motivation of proposing AFRSN P systems is to model weighted fuzzy production rules and the following three types are concerned. AFRSN P systems models for weighted fuzzy production rules are summarized as follows, as shown in Figure 3.

Type 1 R_i: IF p_1 THEN p_2 $(CF = \gamma)$, τ, ω;

Type 2 R_i: IF p_1 AND p_2 AND $\cdots$ AND p_n THEN p_{n+1} $(CF = \gamma)$, τ, $\omega_1, \ldots, \omega_n$;

Type 3 R_i: IF p_1 OR p_2 OR $\cdots$ OR p_n THEN p_{n+1} $(CF = \gamma)$, τ, $\omega_1, \ldots, \omega_n$.

According to dynamic firing mechanism of AFRSN P systems, the computing ways of spikes for AFRSN P systems of these fuzzy production rules are described as follows [20].

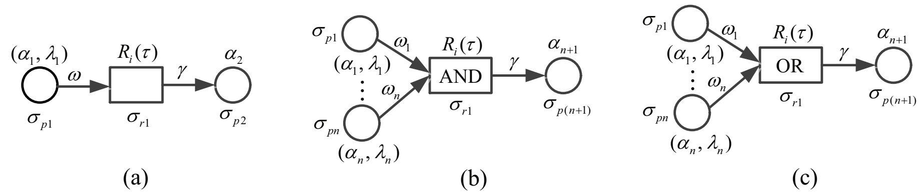


Figure 3: AFRSN P systems of weighted fuzzy production rules of three types (a) type 1. (b) type 2. (c) type 3.

- Type 1: $\alpha_2 = \begin{cases} \alpha_1 \cdot \gamma, & \text{if } \alpha_1 \geq \tau \\ 0, & \text{if } \alpha_1 < \tau \end{cases}$

- Type 2: $\alpha_{n+1} = \begin{cases} \left(\sum\limits_{i=1}^{n} \alpha_i \cdot \omega_i\right) \cdot \gamma, & \text{if } \left(\sum\limits_{i=1}^{n} \alpha_i \cdot \omega_i\right) \geq \tau \\ 0, & \text{if } \left(\sum\limits_{i=1}^{n} \alpha_i \cdot \omega_i\right) < \tau \end{cases}$

- Type 3: $\alpha_{n+1} = \begin{cases} \max(\alpha_j \cdot \gamma_j), & \text{if } \alpha_j \geq \tau_j, j \in J \\ 0, & \text{if } \alpha_j < \tau_j, j = 1, 2 \ldots, n \end{cases}$

WFRSN P systems

Weighted fuzzy reasoning spiking neural P systems (WFRSN P systems) were first introduced in [18] and further investigated in [19]. The definition of a WFRSN P system is described in Definition 4.

Definition 4. A WFRSN P system of degree $m \geq 1$ is a tuple $\Pi = (O, \sigma_1, \ldots, \sigma_m, \mathit{syn}, \mathit{in}, \mathit{out})$, where:

1. $O = \{a\}$ is a singleton alphabet (a is called spike);
2. $\sigma_1, \ldots, \sigma_m$ are neurons, of the form $\sigma_i = (\theta_i, c_i, \overrightarrow{\omega_i}, \lambda_i, r_i), 1 \leq i \leq m$, where:
   (a) θ_i is a real number in [0, 1] representing the potential value of spikes (i.e. value of electrical impulses) contained in neuron σ_i;
   (b) c_i is a real number in [0, 1] representing the truth value associated with neuron σ_i;
   (c) $\overrightarrow{\omega_i} = (\omega_{i1}, \ldots, \omega_{iN_i})$ is a real number vector in (0, 1] representing the output weight vector of neuron σ_i, where ω_{ij} ($1 \leq j \leq N_i$) represents the weight on jth output arc (synapse) of neuron σ_i and N_i is a natural number representing the number of synapses starting from neuron σ_i.
   (d) λ_i is a real number in [0, 1) representing the firing threshold of neuron σ_i;
   (e) r_i represents a firing (spiking) rule contained in neuron σ_i with the form $E/a^\theta \to a^\beta$, where θ and β are real numbers in [0, 1], $E = \{a^n, \theta \geq \lambda_i\}$ is the firing condition. The firing condition means that if and only if neuron σ_i receives at least n spikes and the potential value of spikes is with $\theta \geq \lambda_i$, then the firing rule contained in the neuron can be applied, otherwise, the firing rule cannot be applied;
3. $\mathit{syn} \subseteq \{1, \ldots, m\} \times \{1, \ldots, m\}$ with $i \neq j$ for all $(i, j) \in \mathit{syn}, 1 \leq i, j \leq m$; that is, syn provides a (weighted) directed graph whose set of nodes is $\{1, \ldots, m\}$;

4. $in, out \subseteq \{1, \ldots, m\}$ indicate the input neuron set and the output neuron set of Π, respectively.

In what follows, we describe fault diagnosis production rules and their WFRSN P system models, as shown in Figure 4 [19], [49].

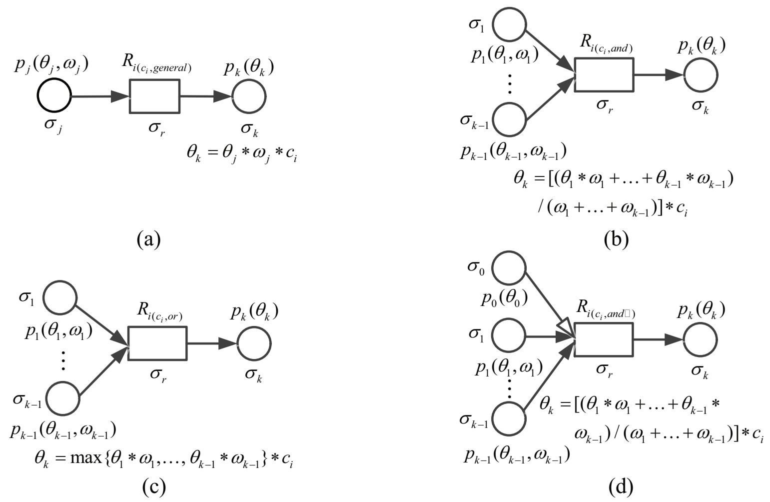


Figure 4: WFRSN P system models for fault diagnosis production rules in TPSSs. (a) *Type 1*; (b) *Type 2*; (c) *Type 3*; (d) *Type4*.

Type 1 (*Simple Rules*) R_i: IF $p_j(\theta_j)$ THEN $p_k(\theta_k)$ (CF $= c_i$), where p_j and p_k are propositions, c_i is a real number in [0, 1] representing the certainty factor of rule R_i, θ_j and θ_k are real numbers in [0, 1] representing the truth values of p_j and p_k, respectively. The weight of proposition p_j is ω_j, where $\omega_j = 1$ because there is only one proposition in the antecedent of this kind of rules. The truth values of p_k is $\theta_k = \theta_j * \omega_j * c_i = \theta_j * c_i$.

Type 2 (*Compound And Rules*) R_i: IF $p_1(\theta_1)$ and $\ldots$ and $p_{k-1}(\theta_{k-1})$ THEN p_k (θ_k) (CF $= c_i$), where $p_1, \ldots, p_k$ are propositions, c_i is a real number in [0, 1] representing the certainty factor of rule R_i, $\theta_1, \ldots, \theta_k$ are real numbers in [0, 1] representing the truth values of $p_1, \ldots, p_k$, respectively. The weights of propositions $p_1, \ldots, p_{k-1}$ are $\omega_1, \ldots, \omega_{k-1}$, respectively. The truth values of p_k is $\theta_k = [(\theta_1 * \omega_1 + \ldots + \theta_{k-1} * \omega_{k-1})/(\omega_1 + \ldots + \omega_{k-1})] * c_i$.

Type 3 (*Compound Or Rules*) R_i: IF $p_1(\theta_1)$ or $\ldots$ or $p_{k-1}(\theta_{k-1})$ THEN p_k (θ_k) (CF $=$ c_i), where $p_1, \ldots, p_k$ are propositions, c_i is a real number in [0, 1] representing the certainty factor of rule R_i, $\theta_1, \ldots, \theta_k$ are real numbers in [0, 1] representing the truth values of $p_1, \ldots, p_k$, respectively. The weights of propositions $p_1, \ldots, p_{k-1}$ are $\omega_1, \ldots, \omega_{k-1}$, respectively. The truth values of p_k is $\theta_k = max\{\theta_1 * \omega_1, \ldots, \theta_{k-1} * \omega_{k-1}\} * c_i$.

Type 4 (*Conditional And Rules*) R_i: WHEN $p_0(\theta_0)$ is true, IF $p_1(\theta_1)$ and $\ldots$ and $p_{k-1}(\theta_{k-1})$ THEN $p_k(\theta_k)$ (CF $= c_i$), where $p_0, \ldots, p_k$ are propositions, c_i is a real number in [0, 1] representing the certainty factor of rule R_i, $\theta_0, \ldots, \theta_k$ are real numbers in [0, 1] representing the truth values of $p_0, \ldots, p_k$, respectively. The proposition p_0 is used to judge whether the reasoning condition of rule R_i is satisfied and its truth value θ_0 is not used in reasoning process. Thus, the weight of θ_0 is not considered in the model. The weights of propositions $p_1, \ldots, p_{k-1}$ are $\omega_1, \ldots, \omega_{k-1}$, respectively. The truth values of p_k is $\theta_k = [(\theta_1 * \omega_1 + \ldots + \theta_{k-1} * \omega_{k-1})/(\omega_1 + \ldots + \omega_{k-1})] * c_i$.

tFRSN P systems

The first version of fuzzy reasoning spiking neural P systems with trapezoidal fuzzy numbers (tFRSN P systems) was introduced in [22] and further expanded in [23]- [25]. The definition of a tFRSN P system is described in Definition 5.

Definition 5. A tFRSN P system with trapezoidal fuzzy numbers of degree $m \geq 1$ is a tuple $\Pi = (O, \sigma_1, \ldots, \sigma_m, syn, in, out)$, where:

1. $O = \{a\}$ is a singleton alphabet (a is called spike);
2. $\sigma_1, \ldots, \sigma_m$ are neurons of the form $\sigma_i = (\theta_i, c_i, r_i), 1 \leq i \leq m$, where
   (a) θ_i is a trapezoidal fuzzy number in [0, 1] representing the potential value of spikes (i.e., the value of electrical impulses) contained in neuron σ_i;
   (b) c_i is a trapezoidal fuzzy number in [0, 1] representing the fuzzy truth value corresponding to neuron σ_i;
   (c) r_i represents a firing (spiking) rule associated with neuron σ_i of the form $E/a^{\theta} \rightarrow a^{\beta}$, where θ and β are trapezoidal fuzzy numbers in [0, 1], $E = a^s$ is the firing condition, it means that the spiking rule contained in neuron σ_i, can be applied if and only if neuron σ_i contains at least s spikes, otherwise, the firing rule cannot be applied;
3. $syn \subseteq \{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\}$ with $i \neq j$ for all $(i, j) \in syn, 1 \leq i, j \leq m$, is a directed graph of synapses between the linked neurons;
4. $in, out \in \{1, 2, \ldots, m\}$ indicate the input neuron set and the output neuron set of Π, respectively.

A *trapezoidal fuzzy number* can be parameterized by a 4-tuple $\tilde{A}=(a_1, a_2, a_3, a_4)$, as shown in Fig. 5, where a_1, a_2, a_3 and a_4 are real numbers such that $a_1 < a_2 < a_3 < a_4$, which are the four horizontal axis values of the trapezoid. The membership function $\mu_{\tilde{A}}(x)$ of the trapezoidal fuzzy number $\tilde{A}$ is defined as follows.

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 < x \leq a_2 \\ 1, & a_2 < x \leq a_3 \\ \frac{a_4-x}{a_4-a_3}, & a_3 < x \leq a_4 \\ 0, & x > a_4 \end{cases} \tag{6}$$

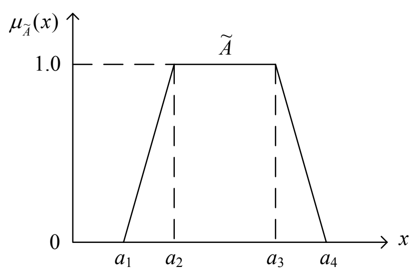


Figure 5: A trapezoidal fuzzy number.

Let $\tilde{A}$ and $\tilde{B}$ be two trapezoidal fuzzy numbers, $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$. The arithmetic operations of the trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are listed as follows. More operations can be seen in [25], [29].

1. Addition $\oplus$: $\tilde{A} \oplus \tilde{B} = (a_1, a_2, a_3, a_4) \oplus (b_1, b_2, b_3, b_4) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$;
2. Subtraction $\ominus$: $\tilde{A} \ominus \tilde{B} = (a_1, a_2, a_3, a_4) \ominus (b_1, b_2, b_3, b_4) = (a_1 - b_1, a_2 - b_2, a_3 - b_3, a_4 - b_4)$;
3. Multiplication $\otimes$: $\tilde{A} \otimes \tilde{B} = (a_1, a_2, a_3, a_4) \otimes (b_1, b_2, b_3, b_4) = (a_1 \times b_1, a_2 \times b_2, a_3 \times b_3, a_4 \times b_4)$;
4. Division $\oslash$: $\tilde{A} \oslash \tilde{B} = (a_1, a_2, a_3, a_4) \oslash (b_1, b_2, b_3, b_4) = (a_1/b_1, a_2/b_2, a_3/b_3, a_4/b_4)$.

Four logic operations are list as follows, where A and B are trapezoidal fuzzy numbers, and a, b are real numbers [25].

1. *Minimum operator* $\wedge$: $a \wedge b = min(a, b)$;
2. *Maximum operator* $\vee$: $a \vee b = max(a, b)$;
3. *and* $\textcircled{\wedge}$: $A \textcircled{\wedge} B = (a_1, a_2, a_3, a_4) \textcircled{\wedge} (b_1, b_2, b_3, b_4) = ((a_1 \wedge b_1), (a_2 \wedge b_2), (a_3 \wedge b_3), (a_4 \wedge b_4))$;
4. *or* $\textcircled{\vee}$: $A \textcircled{\vee} B = (a_1, a_2, a_3, a_4) \textcircled{\vee} (b_1, b_2, b_3, b_4) = ((a_1 \vee b_1), (a_2 \vee b_2), (a_3 \vee b_3), (a_4 \vee b_4))$.

In addition, a scalar multiplication operation is list as follows, where A is a trapezoidal fuzzy number and b is a real number [25].

Scalar Multiplication: $bA = b(a_1, a_2, a_3, a_4) = (ba_1, ba_2, ba_3, ba_4)$.

In what follows, we summarize tFRSN P system models for rules *Type 1* to *Type 4*, as shown in Figure 6. In the following description, R_i $(i = 1, \ldots, N_r)$ is the ith fuzzy production rule, N_r represents the number of fuzzy production rules, c_i is a trapezoidal fuzzy number in [0, 1] representing the certainty factor of R_i, p_j $(1 \leq j \leq N_p)$ is the jth proposition appearing in the antecedent or consequent part of R_i, N_p represents the number of proportions, and θ_j is a trapezoidal fuzzy number in [0, 1] representing the fuzzy truth value of proposition p_j.

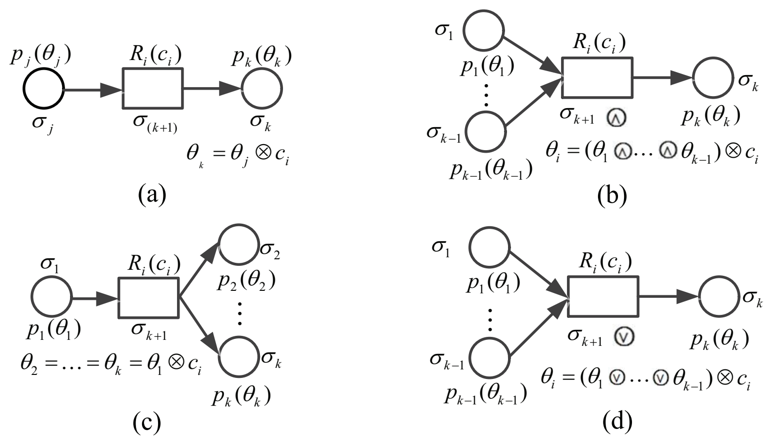


Figure 6: tFRSN P system models for fuzzy production rules. (a) *Type 1*; (b) *Type 2*; (c) *Type 3*; (d) *Type 4*.

Type 1: $R_i(c_i) : p_j(\theta_j) \rightarrow p_k(\theta_k)$ $(1 \leq j, k \leq N_p)$. The fuzzy truth value of the proposition p_k is $\theta_k = \theta_j \otimes c_i$.

Type 2: $R_i(c_i) : p_1(\theta_1) \bigcirc\!\!\!\!\wedge \ldots \bigcirc\!\!\!\!\wedge\ p_{k-1}\ (\theta_{k-1}) \rightarrow p_k\ (\theta_k)$. The fuzzy truth value of the proposition p_k is $\theta_k = (\theta_1 \bigcirc\!\!\!\!\wedge \ldots \bigcirc\!\!\!\!\wedge \theta_{k-1}) \otimes c_i$.

Type 3: $R_i(c_i) : p_1(\theta_1) \rightarrow p_2(\theta_2) \bigcirc\!\!\!\!\wedge \ldots \bigcirc\!\!\!\!\wedge\ p_k\ (\theta_k)$. The fuzzy truth values of the propositions $p_2, p_3, \ldots, p_k$ are identical, i.e., $\theta_2 = \theta_3 = \ldots = \theta_k = \theta_1 \otimes c_i$.

Type 4: $R_i(c_i) : p_1(\theta_1) \bigcirc\!\!\!\!\vee \ldots \bigcirc\!\!\!\!\vee\ p_{k-1}\ (\theta_{k-1}) \rightarrow p_k(\theta_k)$. The fuzzy truth value of the proposition p_k is $\theta_k = (\theta_1 \bigcirc\!\!\!\!\vee \ldots \bigcirc\!\!\!\!\vee \theta_{k-1}) \otimes c_i$.

The fuzzy truth values of these propositions appearing in the fuzzy production rules and the certainty factor of each fuzzy production rule can also be described by using linguistic terms, which are represented by the trapezoidal fuzzy numbers shown in Table 1.

Linguistic Terms	Trapezoidal Fuzzy Numbers
absolutely-false (AF)	(0, 0, 0, 0)
very-low (VL)	(0, 0, 0.02, 0.07)
low (L)	(0.04, 0.1, 0.18, 0.23)
medium-low (ML)	(0.17, 0.22, 0.36, 0.42)
medium (M)	(0.32, 0.41, 0.58, 0.65)
medium-high (MH)	(0.58, 0.63, 0.80, 0.86)
high (H)	(0.72, 0.78, 0.92, 0.97)
very-high (VH)	(0.975, 0.98, 1, 1)
absolutely-high (AH)	(1, 1, 1, 1)

Table 1: Linguistic terms and their corresponding trapezoidal fuzzy numbers

2.4 Fuzzy knowledge reasoning

Fuzzy reasoning spiking neural P systems (FRSN P systems) are proposed to handle fuzzy knowledge. In what follows, we will summarize fuzzy reasoning algorithms for the following FRSN P systems classes: rFRSN P systems, AFRSN P systems, WFRSN P systems and tFRSN P systems.

Fuzzy reasoning based on rFRSN P systems

A fuzzy reasoning algorithm (FRA) based on rFRSN P systems was proposed in [12]. The goal of FRA is to reason out the fuzzy truth values of unknown fuzzy propositions (proposition neurons) from known fuzzy propositions (input neurons). These unknown fuzzy propositions are associated with output neurons. Suppose all fuzzy production rules in a fuzzy diagnosis knowledge base have been modeled by an rFRSN P system model Π. The model Π consists of m neurons consisting of n proposition neurons and k rule neurons (AND type neurons and OR type neurons), where $m = n + k$.

A description of an FRA for an rFRSN P system is shown below. For details about the involved parameter vectors, matrices and multiplication operations, please see [12].

INPUT: parameter matrixes U, V, Λ, H_1, H_2, λ_p, λ_r, and initial inputs α_p^0, a_p^0.
OUTPUT: The fuzzy truth values of propositions associated with the neurons in O.
Step 1) Let $\alpha_r^0 = (0, 0, \ldots, 0)^T$, $a_r^0 = (0, 0, \ldots, 0)^T$.
Step 2) Let $t = 0$.
Step 3)

(1) Process the firing of proposition neurons.
$\beta_p^t = \mathit{fire}(\alpha_p^t, a_p^t, \lambda_p)$, $b_p^t = \mathit{fire}(1, a_p^t, \lambda_p)$, $\alpha_p^t = \mathit{update}(\alpha_p^t, a_p^t, \lambda_p)$,
$a_p^t = \mathit{update}(a_p^t, a_p^t, \lambda_p)$, $B_p^t = \mathit{diag}(b_p^t)$.

(2) Compute the truth values of rule neurons and the number of received spikes.
$\alpha_r^{t+1} = \alpha_r^t \oplus [(H_1 \cdot ((B_p^t \cdot U)^T \odot \beta_p^t)) + (H_2 \cdot ((B_p^t \cdot U)^T \otimes \beta_p^t)]$,
$a_r^{t+1} = a_r^t + [(B_p^t \cdot U)^T \cdot b_p^t]$.

(3) Process the firing of rule neurons.
$\beta_r^{t+1}=\mathit{fire}(\Lambda \cdot \alpha_r^{t+1}, a_r^{t+1}, \lambda_r)$, $b_r^{t+1}=\mathit{fire}(1, a_r^{t+1}, \lambda_r)$,
$\alpha_r^{t+1}=\mathit{update}(\alpha_r^{t+1}, a_r^{t+1}, \lambda_r)$, $a_r^{t+1}=\mathit{update}(a_r^{t+1}, a_r^{t+1}, \lambda_r)$, $B_r^{t+1}=\mathit{diag}(b_r^{t+1})$.

(4) Compute the truth values of proposition neurons and the number of received spikes.
$\alpha_p^{t+1} = \alpha_p^t \oplus [(V \cdot B_r^{t+1}) \otimes \beta_r^{t+1}]$, $a_p^{t+1} = a_p^t + [(V \cdot B_r^{t+1}) \cdot b_r^{t+1}]$.

Step 4) If $a_p^{t+1} = (0, 0, \ldots, 0)^T$ and $a_r^{t+1} = (0, 0, \ldots, 0)^T$ (computation halts), the reasoning results are obtained; otherwise, $t = t + 1$, go to Step 3).

Fuzzy reasoning based on AFRSN P systems

The fuzzy reasoning algorithm for AFRSN P systems is about the fuzzy reasoning process of above weighted fuzzy production rules. Let $Pcurrent = \{\sigma_{pi} | \sigma_{pi} \in N_p, \alpha_i > 0\}$ be a set of current enabled proposition neurons. If a neuron $\sigma_{pi} \in Pcurrent$, then it fires. Let $Rcurrent = \{\sigma_{rj} | \sigma_{rj} \in N_r, \alpha_i > \tau_j\}$ be a set of current enabled rule neurons. Likewise, if a neuron $\sigma_{rj} \in Rcurrent$, then it fires. A fuzzy reasoning algorithm for AFRSN P systems can be summarized as follows.

INPUT: Certainty factors of a set of antecedent propositions, which correspond to I of an AFSN P system.
OUTPUT: The fuzzy truth values of propositions associated with the neurons in O.
Step 1) $Pcurrent := I$, $Rcurrent := \{\}$, $P := Np$, $R := Nr$;
Step 2) Let t =0, where t represents the reasoning step;
Step 3) Compute the outputs of current enabled proposition neurons in $Pcurrent$;
Step 4) Find current enabled rule neurons $Rcurrent$ form R;
Step 5) Compute the outputs of current enabled proposition neurons in $Rcurrent$;
Step 6) $P := P - Pcurrent$, $R := R - Rcurrent$;
Step 7) Find current enabled proposition neurons $Pcurrent$ form P;
Step 8) If $P = \{\}$ *or* $R = \{\}$ (computation halts), the reasoning
results are obtained; otherwise, $t = t + 1$, go to Step 3).

A weight learning algorithm for AFRSN P systems is summarized as follows.

INPUT: Training data set D, $m = |D|$, learning rate delta.
OUTPUT: Weights $(w_1, w_2, ..., w_n)$, where n is the number of weights.
Step 1) Select a set of initial weights;
Step 2) Let t =1, where t represents the reasoning step;
Step 3) Update the weights $(w_1, w_2, ..., w_n)$, using Widrow-Hoff learning law with learning rate delta;
Step 4) If $t > m$ (computation halts) and m represents the number of proposition neurons, the reasoning results are obtained; otherwise, $t = t + 1$, go to Step 3).

Fuzzy reasoning based on WFRSN P systems

A weighted matrix fuzzy reasoning algorithm (WMFRA) for WFRSN P systems can be summarized as follows [19]. For details about the involved parameter vectors, matrices and multiplication operations, please see [19].

INPUT: The fuzzy truth values of the propositions corresponding to the input proposition neurons.
OUTPUT: The fuzzy truth values of the propositions corresponding to the output proposition neurons.
Step 1) Let $g = 0$ be the reasoning step;
Step 2) Set initial values of $\boldsymbol{W}_{r1}$, $\boldsymbol{W}_{r2}, \boldsymbol{W}_{r3}$, $\boldsymbol{W}_p$, $\boldsymbol{\lambda}_p$, $\boldsymbol{\lambda}_r$, $\mathbf{C}$, and the termination condition $\mathbf{0}_1 = (0, \overset{(t)}{\ldots}, 0)^T$. The initial values of $\boldsymbol{\theta}$ and $\boldsymbol{\delta}$ are set to $\boldsymbol{\theta}_g = (\theta_{1g}, \theta_{2g}, \ldots, \theta_{sg})$ and $\boldsymbol{\delta}_g = (\delta_{1g}, \delta_{2g}, \ldots, \delta_{tg})$, respectively;
Step 3) g is increased by one;
Step 4) The firing condition of each input neuron ($g = 1$) or each proposition neuron ($g > 1$) is evaluated. If the condition $E = \{a^n, \theta_i \geq \lambda_{pi}, 1 \leq i \leq s\}$ is satisfied and there is a postsynaptic rule neuron, the neuron fires and transmits a spike to the next rule neuron;
Step 5) Compute the fuzzy truth value vector $\boldsymbol{\delta}_g$ according to (7);

$$\boldsymbol{\delta}_{g+1} = (\mathbf{W}_{r1}^T \otimes \boldsymbol{\theta}_g) + (\mathbf{W}_{r2}^T \oplus \boldsymbol{\theta}_g) + (\mathbf{W}_{r3}^T \odot \boldsymbol{\theta}_g) \tag{7}$$

Step 6) If $\boldsymbol{\delta}_g = \mathbf{0}_1$, the algorithm stops and outputs the reasoning results ;

Step 7) Evaluate the firing condition of each rule neuron. If the condition $E = \{a^n, \delta_j \geq \lambda_{rj}, 1 \leq j \leq t\ \}$ holds, the rule neuron fires and transmits a spike to the next proposition neuron;
Step 8) Compute the fuzzy truth value vector $\boldsymbol{\theta}_g$ according to (8). Go to Step 3).

$$\boldsymbol{\theta}_{\boldsymbol{g+1}} = \mathbf{W}_p^T \odot (\mathbf{C} \otimes \boldsymbol{\delta}_{\boldsymbol{g+1}}) \tag{8}$$

Fuzzy reasoning based on tFRSN P systems

A matrix-based fuzzy reasoning algorithm (MBFRA) for tFRSN P systems can be summarized as follows [25]. For details about the involved parameter vectors, matrices and multiplication operations, please see [25].

INPUT: The fuzzy truth values of the propositions corresponding to the input proposition neurons.
OUTPUT: The fuzzy truth values of the propositions corresponding to the output proposition neurons.
Step 1) Let $g = 0$ be the reasoning step;
Step 2) Set initial values of $\boldsymbol{D}_1$, $\boldsymbol{D}_2$, $\boldsymbol{D}_3$, $\boldsymbol{E}$, $\boldsymbol{C}$ and the termination condition $\mathbf{0}_1 = (unknown, \overset{(t)}{\ldots}, unknown)^T$. The initial values of $\boldsymbol{\theta}$ and $\boldsymbol{\delta}$ are set to $\boldsymbol{\theta}_g = (\theta_{1g}, \theta_{2g}, \ldots, \theta_{sg})$ and $\boldsymbol{\delta}_g = (\delta_{1g}, \delta_{2g}, \ldots, \delta_{tg})$, respectively;
Step 3) g is increased by one;
Step 4) The firing condition of each input neuron ($g = 1$) or each proposition neuron ($g > 1$) is evaluated. If the condition $E = a^s$ is satisfied and there is a postsynaptic rule neuron, the neuron fires and transmits a spike to the next rule neuron;
Step 5) Compute the fuzzy truth value vector $\boldsymbol{\delta}_g$ according to (9);

$$\boldsymbol{\delta}_g = (\boldsymbol{D}_1^T \circledcirc \boldsymbol{\theta}_{g-1}) \oplus (\boldsymbol{D}_2^T \odot \boldsymbol{\theta}_{g-1}) \oplus (\boldsymbol{D}_3^T \circledast \boldsymbol{\theta}_{g-1}) \tag{9}$$

Step 6) If $\boldsymbol{\delta}_g = \mathbf{0}_1$, the algorithm halts and outputs the reasoning results;
Step 7) Evaluate the firing condition of each rule neuron. If the condition $E = a^s$ is satisfied,

the rule neuron fires and transmits a spike to the next proposition neuron;
Step 8) Compute the fuzzy truth value vector $\boldsymbol{\theta}_g$ according to (10). Go to Step 3).

$$\theta_g = \boldsymbol{E}^T \circledast (\boldsymbol{C} \circledcirc \delta_g) \tag{10}$$

2.5 Comparisons between SN P systems and FRSN P systems

rFRSN P systems, lFRSN P systems, AFRSN P systems, WFRSN P systems and tFRSN P systems are collectively called FRSN P systems in this paper. FRSN P systems are novel graphical models for representing and reasoning fuzzy knowledge and information. How FRSN P systems are extended from SN P systems is described as follows and comparisons about neurons and spiking rules between SN P systems and FRSN P systems are shown in Table 2. It is worth pointing out that the spiking rule is collectively described with the form $E/a^{\alpha} \to a^{\beta}$ in FRSN P systems for convenient representation.

1. The content of a neuron is the potential value of spikes contained in this neuron instead of the number of spikes in SN P systems;

2. Each neuron in an FRSN P system associates with either a fuzzy proposition or a fuzzy production rule;

3. Each neuron contains only one spiking (firing) rule, with the form $E/a^{\alpha} \to a^{\beta}$, where $E = a^n$ is the firing condition and n represents the number of input synapses from other neurons to this neuron. The firing condition $E = a^n$ indicates that if the neuron receives n spikes, the spiking rule can be applied; otherwise the rule cannot be enabled until n spikes are received;

4. The firing mechanism of neurons in FRSN P systems is described as follows. For the neuron σ_i, if its firing rule $E/a^{\alpha} \to a^{\beta}$ can be applied, then the neuron fires. This means its pulse value $\alpha > 0$ is consumed (removed) and it produces a spike with value β. Once a spike with value β is excited from neuron σ_i, all neurons σ_j with $(i, j) \in syn$ immediately receive the spike;

5. Different types of neurons are defined with different ways to handle spikes;

6. Time delay is ignored in an FRSN P system, thus all neurons are always open;

7. For AFRSN P systems and WFRSN P systems, their synapses have output weights.

3 Applications

This section will outline the application developments of fuzzy membrane computing including fuzzy knowledge representation and fault diagnosis of transformers, traction power supply systems and power transmission networks based on FRSN P systems. The description of the essentials of electrical power system fault diagnosis, please see [25] and principles of model-based fault diagnosis methods, please see [19].

P systems	Neurons			Spiking rules
	Type	Content	Expression	
SN P systems	one	number of spikes	computing space	$E/a^c \to a^p; d$
rFRSN P systems	proposition neuron rule neuron: and, or	pulse value (real numbers)	propositions, fuzzy production rules	$E/a^\alpha \to a^\beta$
lFRSN P systems	proposition neuron rule neuron	pulse value (linguistic terms)	propositions, fuzzy production rules	$E/a^\mu \to a^\nu$
AFRSN P systems	proposition neuron rule neuron	pulse value (real numbers)	propositions, fuzzy production rules	$E/a^\alpha \to a^\beta$
WFRSN P systems	proposition neuron rule neuron: general, and, or	pulse value (real numbers)	propositions, fuzzy production rules	$E/a^\alpha \to a^\beta$
tFRSN P systems	proposition neuron rule neuron: general, and, or	pulse value (trapezoidal fuzzy numbers)	propositions, fuzzy production rules	$E/a^\theta \to a^\beta$

Table 2: Comparisons about neurons and spiking rules between SN P systems and FRSN P systems

3.1 Fuzzy knowledge representation

Several kinds of FRSN P systems were proposed for fuzzy knowledge representation and reasoning. In this subsection, an example of fuzzy knowledge representation based on lFRSN P systems is summarized to show the effectiveness of FRSN P systems in representing fuzzy knowledge. Assume that there are seven fuzzy production rules in a rule set seven propositions represented by $p_1, p_2, p_3, p_4, p_5, p_6, p_7$ [16].

$R_1 : IF\ p_1\ THEN\ p_2\ (CF = almost\ certain)$
$R_2 : IF\ p_2\ THEN\ p_3\ (CF = pretty\ true)$
$R_3 : IF\ p_2\ THEN\ p_4\ (CF = rather\ true)$
$R_4 : IF\ p_1\ THEN\ p_6\ (CF = pretty\ true)$
$R_5 : IF\ p_6\ THEN\ p_4\ and\ p_5\ (CF = very\ true)$
$R_6 : IF\ p_1\ THEN\ p_7\ (CF = almost\ certain)$
$R_7 : IF\ p_7\ THEN\ p_4\ (CF = sort\ of\ true)$

The fuzzy truth value *very true* of proposition p_1 given by a user, and he wants to know the truth value of proposition p_4. The lFRSN P system model of these fuzzy production rules is constructed, as shown in Figure 7, to find the truth value of proposition p_4 which is the potential value of σ_4.

From Figure 7, there are three paths from neuron σ_1 to neuron σ_4 and are described as follows:

$path\ (1) :\ \sigma_1 \to \sigma_8 \to \sigma_7 \to \sigma_{11} \to \sigma_4$
$path\ (2) :\ \sigma_1 \to \sigma_9 \to \sigma_2 \to \sigma_{12} \to \sigma_4$
$path\ (3) :\ \sigma_1 \to \sigma_{10} \to \sigma_6 \to \sigma_{14} \to \sigma_4$

According to computing mechanism of neurons in lFRSN P systems, we get results described as follows: from path (1), potential value of σ_4 is: $very\ true \otimes almost\ certain \otimes sort\ of\ true = sort\ of\ true$; from path (2), potential value of σ_4 is: $very\ true \otimes almost\ certain \otimes rather\ true = rather\ true$; from path (3), potential value of σ_4 is: $very\ true \otimes pretty\ true \otimes very\ true = pretty\ true$. Therefore, potential value of σ_4 is: $sort\ of\ true \oplus rather\ true \oplus pretty\ true =$

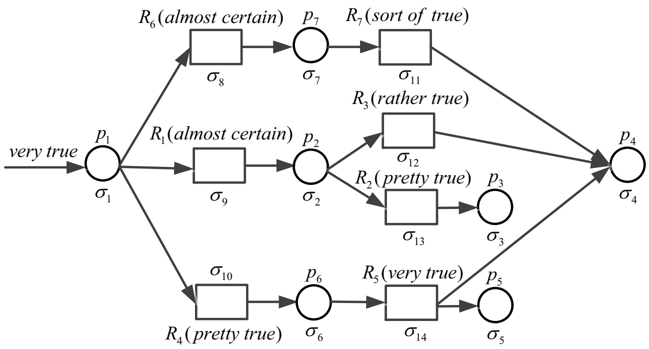


Figure 7: An lFRSN P system model for seven fuzzy production rules.

pretty true. Thus, truth value of proposition is *pretty true*.

3.2 Transformers

In this subsection, an application example is used to demonstrate the effectiveness of rFRSN P systems and their FRA in fault diagnosis of a transformer. The following fuzzy production rules are obtained from the knowledge base of a transformer fault diagnosis system [12].

Rule 1 (CF=0.8)
Symptom:
1) Total hydrocarbon is little high (p_1);
2) C_2H_2 is low (p_2);
Anticipated Fault: General overheating fault occurs (p_{11}).
Rule 2 (CF=0.8)
(1) Total hydrocarbon is rather high (p_3);
(2) C_2H_2 is too high (p_4);
(3) H_2 is high (p_5);
(4) C_2H_2 in total hydrocarbon occupies a too low proportion (p_6);
Anticipated Fault: Serious overheating fault occurs (p_{11}).
Rule 3 (CF=0.8)
(1) Total hydrocarbon is little low (p_7);
(2) H_2 is high (p_5);
(3) CH_4 in total hydrocarbon occupies a large proportion (p_8);
(4) CH_4 in total hydrocarbon occupies a higher proportion than C_2H_2 (p_9);
Anticipated Fault: The partial discharge occurs (p_{13}).
Rule 4 (CF=0.8)
(1) Total hydrocarbon is rather low (p_{10});
(2) C_2H_2 is too high (p_4);
(3) H_2 is high (p_5);
Anticipated Fault: The spark discharge occurs (p_{14}).

These fuzzy production rules can be modeled by the following rFRSN P system Π_5, as shown in Figure 8.

$\Pi_5 = (A, \sigma_1, \ldots, \sigma_{14}, \sigma_{15}, \ldots, \sigma_{18}, syn, I, O)$, where

(1) $A = \{a\}$.

(2) $\sigma_1, \ldots, \sigma_{14}$ are proposition neurons associated with propositions $p_1, \ldots, p_{14}$ respectively.

(3) $\sigma_{15}, \ldots, \sigma_{18}$ are AND-type rule neurons associated with production rules $R_1, \ldots, R_4$ respectively.

(4) $syn = \{(1,15),(2,15),(3,16),(4,16),(4,18),(5,16),(5,17),(5,18),(6,18),$ $(7,17),(8,17),(9,17),(10,18),(15,11),(16,12),(17,13),(18,14)\}$.

(5) $I = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}\}$, $O = \{\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}\}$.

According to the definition of parameter vectors and matrices given in [12], U, V, Λ, H_1 and H_2 are follows:

$$U = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T \qquad H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \qquad H_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0.8 \end{bmatrix}$$

$$\lambda_p = (1,1,1,1,1,1,1,1,1,1,1,1,1,1)^T \qquad \lambda_r = (2,4,4,3)^T$$

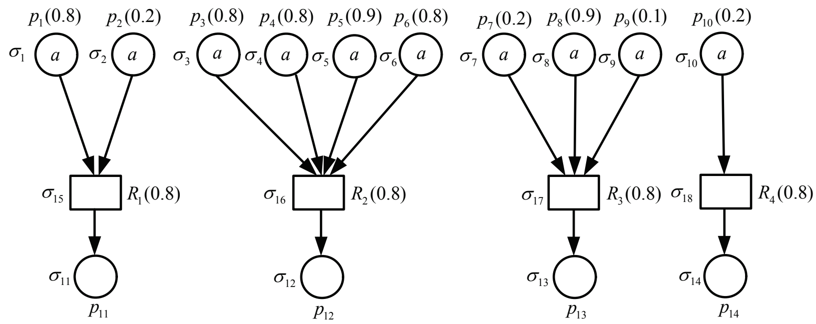


Figure 8: An example of a transformer fault diagnosis modeled by an rFRSN P system model Π_5.

In on-scene information detection of transformer, total hydrocarbon content is high (CF=0.8), C_2H_2 content is high (CF=0.8), H_2 content is high (CF=0.9), C_2H_2 content in total hydrocarbon content is little (CF=0.8), CH_4 content in total hydrocarbon content is little (CF=0.1). Thus, initial truth value vector $\alpha_p^0 = (0.8, 0.2, 0.8, 0.8, 0.9, 0.8, 0.2, 0.9, 0.1, 0.2, 0, 0, 0, 0)^T$ and initial spike vector $a_p^0 = (1,1,1,1,1,1,1,1,1,1,0,0,0,0)^T$. Let $\alpha_r^0 = (0,0,0,0)^T$ and $a_r^0 = (0,0,0,0)^T$.

According to reasoning algorithm described *subsection 2.4*, we get

(1) $\alpha_p^1 = (0,0,0,0,0,0,0,0,0,0,0,0,0,0)^T$, $\alpha_r^1 = (0.16, 0.64, 0.08, 0.16)^T$,
$a_p^1 = (0,0,0,0,0,0,0,0,0,0,0,0,0,0)^T$, $a_r^1 = (2,4,4,3)^T$;
(2) $\alpha_p^2 = (0,0,0,0,0,0,0,0,0,0,0.16,0.64,0.08,0.16)^T$, $\alpha_r^2 = (0,0,0,0)^T$,
$a_p^2 = (0,0,0,0,0,0,0,0,0,0,1,1,1,1)^T$, $a_r^2 = (0,0,0,0)^T$;
(3) $a_p^3 = (0,0,0,0,0,0,0,0,0,0,0,0,0,0)^T$, $a_r^3 = (0,0,0,0)^T$.

Since the system reaches halting computation ($a_p^3 = (0,0,0,0,0,0,0,0,0,0,0,0,0,0)^T$ and $a_r^3 = (0,0,0,0)^T$), system exports its reasoning results, i.e., the truth values of propositions p_{11}, p_{12}, p_{13} and p_{14} are 0.16, 0.64, 0.08 and 0.16, respectively. These reasoning results indicate the following possible faults: general overheating fault (CF=0.16), serious overheating fault (CF=0.64), partial discharge (CF=0.08) and spark discharge (CF=0.16). In the fault diagnosis system, the threshold value of fault occurrence is set to be 0.6. Thus, we can conclude that the transformer shows a serious overheating fault, which is consistent with the actual situation.

3.3 Traction power supply systems

In [19], three cases from the local system of a TPSS chosen in [49], as shown in Figure 9, are considered as examples to manifest the effectiveness of WFRSN P systems in fault diagnosis, where the external transmission lines in a power system which supplies the TPSS are hypothetical, S and R represent the sending end and receiving end of transmission lines, L represents transmission lines. The first two cases are in normal power supply and the third case is in over zone feeding. It is worth pointing out that, the complete line connection of FS1, ATP1, SP1, FS3, ATP3 and TPS-02 is the same as that of TSS-01, FS2, SP2 and ATP2 in Figure 9.

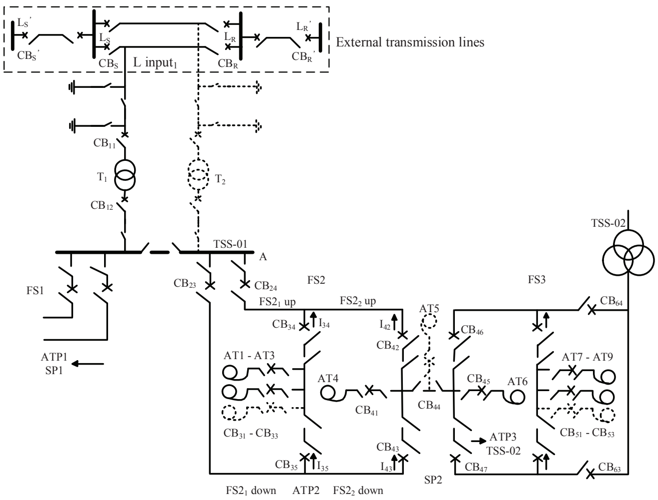


Figure 9: A local single line sketch map of a TPSS.

Case 1: normal power supply. $FS2_{1\ up}$ and AT1 have faults.

Status information from the SCADA system (in time order): $AT1_m$ operated, CB_{31} tripped, $AT3$ auto switched over; $FS2_m$ operated, CB_{23} and CB_{24} tripped; feeder lines auto reclosed, $FS2_{up\ m}$ operated quickly, CB_{23} tripped again. When faults occur, current directions of I_{34} and I_{35} are positive, and current is not detected in SP2.

A WFRSN P system for $FS2_{up}$ is Π_6 and its corresponding WFRSN P system fault diagnosis model is shown in Figure 10.

$\Pi_6 = (O, \sigma_1, \ldots, \sigma_{16}, syn, in, out)$, where:

(1) $O = \{a\}$ is the singleton alphabet (a is called spike);

(2) $\sigma_1, \ldots, \sigma_9$ are proposition neurons corresponding to the propositions with truth values $\theta_1, \ldots, \theta_9$; that is, $s = 9$;

(3) $\sigma_{10}, \ldots, \sigma_{13}$ are rule neurons, where σ_{10}, σ_{11} and σ_{12} are *and* rule neurons, σ_{14} is an *or* rule neuron; that is, $t = 4$;

(4) $syn = \{(1, 10)\ , (2, 10), (2, 11), (3, 11), (4, 12), (5, 12), (6, 13), (7, 13), (8, 13), (10, 6), (11, 7), (12, 8), (13, 9)\}$;

(5) $in = \{\sigma_1, \ldots, \sigma_5\}$, $out = \{\sigma_9\}$.

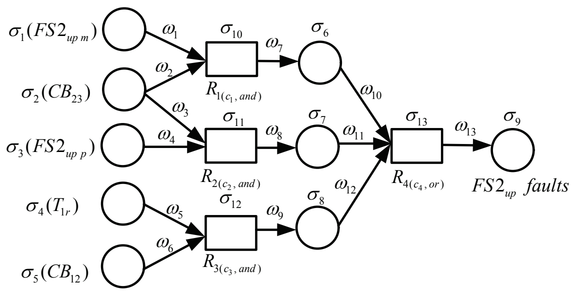


Figure 10: A WFRSN P system fault diagnosis model for $FS2_{up}$.

The synaptic weight matrices of Π_2 are shown in Figure 11 and other parameter matrices associated with the model in Figure 10 are described as follows: $\boldsymbol{\theta_0} = (0.9913\ 0.9833\ 0.8\ 0.4\ 0.2\ 0\ 0\ 0\ 0)^T$, $\boldsymbol{\delta_0} = (0\ 0\ 0\ 0)^T$, $\boldsymbol{C} = diag(0.975\ 0.95\ 0.9\ 0.975)$. In order to succinctly describe the matrices, let us denote $\boldsymbol{O_l} = (x_1, \ldots, x_l)^T$, where $x_i = 0, 1 \leq i \leq l$. When $g = 0$, we get the results: $\boldsymbol{\delta_1} = (0.9873\ 0.8917\ 0.3\ 0)^T$, $\boldsymbol{\theta_1} = (0\ 0\ 0\ 0\ 0\ 0.9626\ 0.8471\ 0.27\ 0)^T$. When $g = 1$, we get the results: $\boldsymbol{\delta_2} = (0\ 0\ 0\ 0.9626)^T$, $\boldsymbol{\theta_2} = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0.9385)^T$. When $g = 2$, we get the results: $\boldsymbol{\delta_3} = (0\ 0\ 0\ 0)^T$. Thus, the termination condition is satisfied and the reasoning process ends. We obtain the reasoning results, i.e., the truth value 0.9385 of the output neuron σ_9. The feeding section $FS2_{up}$ has a fault with a fault confidence level 0.9385. The fault region of $FS2_{up}$ can be further identified according to the fault current detected and the WFRSN P system fault diagnosis model for fault region identification in Figure 12, and then we get the result that $FS2_{1\ up}$ has a fault with a fault confidence level 0.9385.

For AT1, a WFRSN P system can be constructed in a similar way and its corresponding WFRSN P system fault diagnosis model is shown in Figure 13. The diagnosis process of AT1 is similar. According to the SCADA data and Table 3, the parameter matrices of WFRSN P system fault diagnosis model for AT1 is established to perform WMBRA. After the reasoning, the fault confidence level of AT1 is obtained, i.e., 0.8361. So the autotransformer AT1 has a fault with a fault confidence level 0.8361.

Case 2: normal power supply. $FS2_{1\ up}$ *has faults.*

$$\boldsymbol{W_{r1}} = [\mathbf{O}]_{9\times 4}, \boldsymbol{W_{r2}} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{W_{r3}} = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&1 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}, \boldsymbol{W_{p}} = \begin{bmatrix} 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}.$$

Figure 11: Synaptic weight matrices of WFRSN P system fault diagnosis model for $FS2_{up}$.

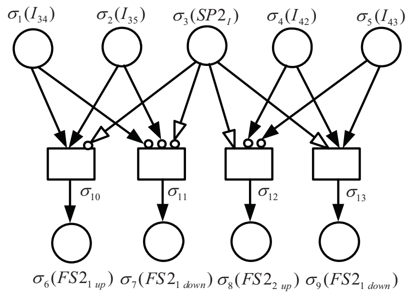


Figure 12: A WFRSN P system fault diagnosis model for fault region identification of a feeding section.

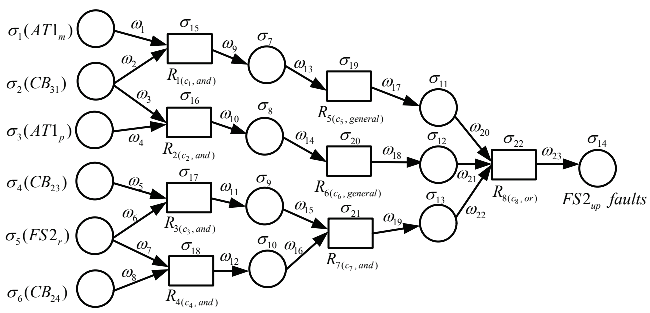


Figure 13: A WFRSN P system fault diagnosis model for AT1.

Sections	Protective devices (operated)						Protective devices (non-operated)					
	Main		Primary backup		Remote backup		Main		Primary backup		Remote backup	
	Relays	CBs	Relays	CBs	Relays	CBs	Relays	CBs	Relays	CBs	Relays	CBs
FL	0.9913	0.9833	0.8	0.85	0.7	0.75	0.2	0.2	0.2	0.2	0.2	0.2
B	0.8564	0.9833	-	-	0.7	0.75	0.4	0.2	-	-	0.4	0.2
T	0.7756	0.9833	0.75	0.8	0.7	0.75	0.4	0.2	0.4	0.2	0.4	0.2

Table 3: Operation and non-operation confidence levels of the protective devices

Status information from the SCADA system (in time order): $FS2_m$ operated, CB_{24} tripped; T_{1r} operated, CB_{11} and CB_{12} tripped. When faults occur, current directions of I_{34} and I_{35} are positive, and current is not detected in SP2. In this case, CB_{23} refused operation.

According to the SCADA data and Table 3, the WFRSN P system fault diagnosis model for

$FS2_1$ and its parameter matrices are established to perform WMBRA. After the reasoning, the fault confidence level of $FS2_{up}$ is obtained, i.e., 0.7439. The fault region of $FS2_{up}$ can be further identified according to the fault current detected and the WFRSN P system fault diagnosis model for fault region identification in Figure 12, and then we get the result that $FS2_{1\ up}$ has a fault. So the feeding section $FS2_{1\ up}$ has a fault with a fault confidence level 0.7439.

Case 3: FS2 is over zone fed by TPS-02. AT7 and $FS2_{2\ up}$ *have faults.*

Status information from the SCADA system (in time order): primary backup protections of feeder lines in SP2 operated, CB_{42} tripped; meanwhile, CB_{51} tripped, AT9 auto switched over; remote backup protection $FS3_s$ of feeder lines in TSS-02 operated, CB_{63} and CB_{64} tripped. When faults occur, current directions of I_{34} and I_{35} are positive, and current is detected only in SP2 and ATP2. In this case, main protection of feeder lines in SP2, CB_{43} and main protection of AT7 refused operation, and status information of primary backup protection of AT7 lost.

According to the SCADA data and Table 3, the WFRSN P system fault diagnosis models for AT7 and $FS2_2$ and their parameter matrices are established to perform WMBRA, respectively. After the reasoning, the fault confidence levels AT7 and $FS2_{up}$ are obtained, i.e., 0.6946 and 0.6123. The fault region of $FS2_{up}$ can be further identified according to the fault current detected and the WFRSN P system fault diagnosis model for fault region identification in Figure 12, and then we get the result that $FS2_{2\ up}$ has a fault. So the autotransformer AT2 has a fault with a fault confidence level 0.6946 and the feeding section $FS2_{2\ up}$ has a fault with a fault confidence level 0.6123.

The results of *Cases 1-3* give evidence of that the proposed fault diagnosis approach can obtain satisfying results both in the situation in normal power supply and over zone feeding with complete/incomplete alarm information. In addition, the proposed method can provide results comparable with those in [49] by using only one simple reasoning while the method in [49] needs a second reasoning.

3.4 Power transmission networks

AFRSN P systems

In this section, four cases of the local system in an EPS shown in Figure 14 are considered as examples to show the effectiveness and superiority of AFRSN P systems for fault diagnosis of power transmission networks. These cases include a single and multiple fault situations [21]. The status information (with/without incompleteness and uncertainty) about protective relays and CBs, and diagnosis results based on AFRSN P systems are shown in Table 4, where " $*$ " means that this case includes incomplete or uncertain status information from the SCADA system. According to Table 4, we know that the fault diagnosis models based on AFSN P systems can deal with the uncertainty of action messages about protective relays and breakers. Furthermore, fault element is diagnosed properly while the information is incomplete because of the well fault tolerance. Therefore, this method is effective in fault diagnosis.

tFRSN P systems

Fault diagnosis method based on tFRSN P systems is called FDSNP for short [25]. This subsection summarizes the FDSNP method, whose flowchart is shown in Figure 15, as follows.

Step 1 Read operation messages about protective relays and/or CBs in a power transmission network from the SCADA system.

Step 2 Search for outage areas. We suggest network topology analysis because it decreases the number of candidate diagnosing areas and reduce the subsequent computational workload [50]. The search process is described as follows:

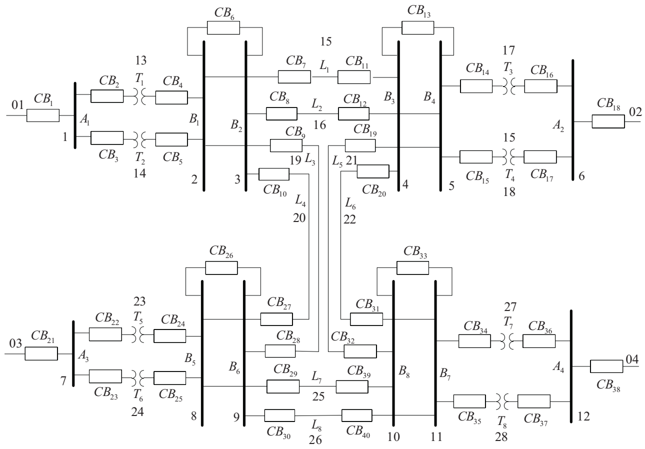


Figure 14: A local sketch map of the protection system of an EPS.

Cases	Status information		Diagnosis results	
	Operated relays	Tripped CBs	Fault section	Fuzzy truth value
1	B_{1m}	$CB_4, CB_5, CB_7, C9, CB_{12}, CB_{27}$	B_1	0.87386
2	B_{1m}, L_{2Rs}, L_{4Rs}	$CB_4, CB_5, CB_7, C9, CB_{12}, CB_{27}$	B_1	0.78044
3	B_{1m}, L_{1Sp}, L_{1Rm}	$CB_4, CB_5, CB_6, CB_7, CB_9, CB_{11}$	B_1, L_1	0.87386, 0.86085
4*	L_{2Rs}, L_{4Rs}	$CB_4, CB_5, CB_7, CB_9, CB_{12}, CB_{27}$	B_1	0.56363

Table 4: Status information and diagnosis results based on AFRSN P systems

(i) Let the search iteration $t = 1$;

(ii) Construct a set Q_t of section numbers: assign a number to each section in the power transmission network. The numbers of all sections constitute the set Q_t;

(iii) Construct a subset M_t of section numbers: put the number of a randomly chosen section from Q_t into the subset M_t. If there is a closed CB connecting this chosen section, find all the closed CBs connecting it, otherwise, go to *(vi)*. Find all the other sections linking with each of the closed CBs and put their numbers from Q_t into M_t. Continue to find the closed CBs and sections according to those in M_t;

(iv) t is increased by one;

(v) Construct the set Q_t: remove the numbers of the sections in M_t from Q_{t-1} and obtain Q_t. If Q_t is not empty, the search process goes to *(iii)*;

(vi) Find passive networks, i.e., outage areas, from $M_1, M_2, \ldots, M_{N_s}$, where N_s is the maximum number of all numbers referring to section subsets. The search process stops.

Step 3 If there is only one section in the passive networks found in *Step 2*, this section is the faulty one and the algorithm stops, otherwise, a fault diagnosis model based on an tFRSN P system is built for each section. The model-building process is described as follows. A section in the passive network is chosen randomly. According to the relay protections of the section, we design fault fuzzy production rules and then determine proposition and rule neurons and create their linking relationship to obtain the tFRSN P system. The certainty factor of each rule is empirically set. According to Tables 5 and 6, we set confidence levels for main protections, first backup protections, second backup protections and their CBs. Then a one-to-one relationship between the fuzzy truth value of each input neuron and the confidence level of each protection is established to obtain the initial values of the model.

Sections	Protective devices					
	Main		First backup		Second backup	
	Relays	CBs	Relays	CBs	Relays	CBs
L	VH	VH	H	H	MH	MH
B	VH	VH	-	-	MH	MH
T	VH	VH	H	H	MH	MH

Table 5: Confidence levels of the operated protective devices

Sections	Protective devices					
	Main		First backup		Second backup	
	Relays	CBs	Relays	CBs	Relays	CBs
L	L	L	L	L	L	L
B	ML	L	-	-	ML	L
T	ML	L	ML	L	L	L

Table 6: Confidence levels of the non-operate protective devices

Step 4 The algebraic fuzzy reasoning algorithm is used to acquire the fault confidence level of each section.

Step 5 If the confidence level θ of a section satisfies the condition $\theta \geq (0.58, 0.63, 0.80, 0.86)$, the section is faulty, otherwise, if θ satisfies the condition $\theta \leq (0.17, 0.22, 0.36, 0.42)$, the section is not faulty, otherwise, the section may be faulty.

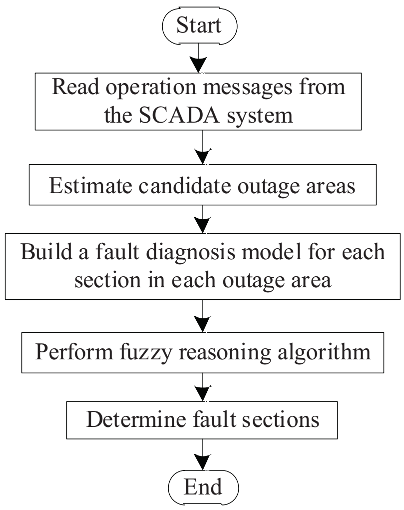


Figure 15: The flowchart of FDSNP.

In [25], seven cases of the local system in an EPS shown in Figure 14 are considered as examples to test the effectiveness and superiority of FDSNP. These cases include single and multiple fault situations. The status information (with/without incompleteness and uncertainty) about protective relays and CBs is shown in Table 7, where " $*$ " means that this case includes incomplete or uncertain status information from the SCADA system.

FDSNP is used to diagnose faults for the seven cases, and the diagnosis results are shown in Table 8, which contains the faulty sections and their fault confidence levels. Table 8 lists the

Cases	Status information	
	Operated relays	Tripped CBs
1	B_{1m}, L_{2Rs}, L_{4Rs}	$CB_4, CB_5, CB_7, CB_9, CB_{12}, CB_{27}$
2*	L_{2Rs}, L_{4Rs}	$CB_4, CB_5, CB_7, CB_9, CB_{12}, CB_{27}$
3	B_{1m}, L_{1Sp}, L_{1Rm}	$CB_4, CB_5, CB_6, CB_7, CB_9, CB_{11}$
4	$B_{1m}, L_{1Sm}, L_{1Rp}, B_{2m}, L_{2Sp}, L_{2Rm}$	$CB_4, CB_5, CB_6, CB_7, CB_8, CB_9, CB_{10}, CB_{11}, CB_{12}$
5	T_{3p}, L_{7Sp}, L_{7Rp}	$CB_{14}, CB_{16}, CB_{29}, CB_{39}$
6	$L_{1Sm}, L_{1Rp}, L_{2Sp}, L_{2Rp}$ $L_{7Sp}, L_{7Rm}, L_{8Sm}, L_{8Rm}$	$CB_7, CB_8, CB_{11}, CB_{12}$ $CB_{29}, CB_{30}, CB_{39}, CB_{40}$
7*	$T_{7m}, T_{8P}, B_{7m}, B_{8m}, L_{5Sm}$ $L_{5Rp}, L_{6Ss}, L_{7Sp}, L_{7Rm}, L_{8Ss}$	$CB_{19}, CB_{20}, CB_{29}, CB_{30}, CB_{32}$ $CB_{33}, CB_{34}, CB_{35}, CB_{36}, CB_{37}, CB_{39}$

Table 7: Status information about protective relays and CBs

Cases	Diagnosis results of FDSNP		
	Fault sections	Fault confidence levels	
		Trapezoidal fuzzy numbers	Linguistic terms
1	B_1	(0.975, 0.98, 1, 1)	VH
2	B_1	(0.5655, 0.6174, 0.80, 0.86)	[M, MH]
3	B_1, L_1	(0.975, 0.98, 1, 1), (0.9506, 0.9604, 1, 1)	VH, [H, VH]
4	B_1	(0.975, 0.98, 1, 1)	VH
	B_2	(0.975, 0.98, 1, 1)	VH
	L_1	(0.9506, 0.9604, 1, 1)	[H, VH]
	L_2	(0.9506, 0.9604, 1, 1)	[H, VH]
5	T_3	(0.72, 0.78, 0.92, 0.97)	H
	L_7	(0.9506, 0.9604, 1, 1)	[H, VH]
6	L_1	(0.702, 0.7644, 0.92, 0.97)	[H, VH]
	L_2	(0.702, 0.7644, 0.92, 0.97)	[H, VH]
	L_7	(0.702, 0.7644, 0.92, 0.97)	[H, VH]
	L_8	(0.9506, 0.9604, 1, 1)	[H, VH]
7	L_5	(0.702, 0.7644, 0.92, 0.97)	[H, VH]
	L_7	(0.702, 0.7644, 0.92, 0.97)	[H, VH]
	B_7	(0.975, 0.98, 1, 1)	[H, VH]
	B_8	(0.975, 0.98, 1, 1)	[H, VH]
	T_7	(0.975, 0.98, 1, 1)	[H, VH]
	T_8	(0.72, 0.78, 0.92, 0.97)	H

Table 8: Fault sections and their fault confidence levels obtained by using FDSNP

fault confidence levels of only faulty sections. Four diagnosis methods, fuzzy logic (FL) [51], fuzzy Petri nets (FPN) [32], genetic algorithm-tabu search (GATS) [52] and genetic algorithm (GA) [53], are used as benchmarks to perform comparative experiments. The diagnosis results of the five methods are shown in Table 9, where " − " means that this case was not considered in the corresponding reference.

From Table 8, we can see that the fault confidence levels represented by trapezoidal fuzzy numbers provide a quantitative description for the faulty sections which makes these results

Cases	Diagnosis results				
	FDSNP	FL [51]	FPN [32]	GATS [52]	GA [53]
1	B_1	B_1	B_1	-	B_1
2	B_1	-	-	-	-
3	B_1, L_1	B_1, L_1	B_1, L_1	-	B_1, L_1
4	B_1, B_2, L_1, L_2	B_1, B_2, L_1, L_2	B_1, B_2, L_1, L_2	-	B_1, B_2, L_1, L_2
5	T_3, T_7	T_3, T_7	T_3, T_7	T_3, T_7	(1)T_3, L_7; (2)T_3 (3)L_7; (4)No
6	L_1, L_2 L_7, L_8	L_1, L_2 L_7, L_8	L_1, L_2 L_7, L_8	L_1, L_2 L_7, L_8	(1)L_1, L_2, L_7, L_8 (2)L_1, L_7, L_8
7	L_5, L_7, B_7 B_8, T_7, T_8	L_5, L_7, B_8 T_7, T_8	L_5, L_7, B_7 B_8, T_7, T_8, L_8	L_5, L_7, B_7 B_8, T_7, T_8	(1)$L_5, L_7, B_7, B_8, T_7, T_8$ (2)L_5, L_7, T_7, B_8

Table 9: Comparisons between FDSNP and four fault diagnosis methods

more reliable. The linguistic terms corresponding to these trapezoidal fuzzy numbers provide a more intuitive and flexible way for experts and dispatchers than probability values, since their knowledge usually contain linguistic terms with a certain degree of uncertainty.

From Table 9, we can see that the diagnosis results of FDSNP, in *Case 1* and *Cases 3-6*, are the same as those in [32, 51], in other words, FDSNP is effective in fault diagnosis of power transmission networks in power systems. Table 9 also shows that, in some cases, FDSNP is superior to FL, FPN and GA on correctly identifying fault sections. For instance, in *Case 7*, the fault diagnosis result of FDSNP is different from those in [32, 51, 53]. In this case, for section L_8, only its second backup protective relay S_{L8Ss} operated and actually, S_{L8Ss} operated as the second backup protective relay of section B_8. So in fact, L_8 is not a faulty section. For section B_7, its main protective relay B_{7m} operated and tripped its corresponding CBs, CB_{33}, CB_{34} and CB_{35}. So B_7 is a faulty section. Thus, for *Case 7*, the diagnosis result of FDSNP and GATS is better than those in [32, 51, 53]. In *Cases 5-7*, comparisons of diagnosis results between FDSNP and the methods in [53] show that FDSNP can solve the nonuniqueness problem of the diagnosis solution, which proves the correctness of FDSNP in diagnosing faulty sections. Besides, the diagnosis results in *Cases 2* and *7* show that FDSNP can obtain satisfying results in the situations with incomplete or uncertain alarm information. Therefore, from the seven typical cases, FDSNP is effective with a good accuracy in fault diagnosis of power transmission networks.

4 Conclusions and future research lines

Fuzzy membrane computing is an important research branch of membrane computing. Until now there have been two main fuzzy P systems: fuzzy cell-like P systems and fuzzy reasoning spiking neural P systems. In this survey, the theoretical developments and applications about fuzzy membrane computing are summarized. To advance this research direction, we list some promising topics as follows:

(1) Extensions and applications of fuzzy cell-like P systems. From the survey in Sections 2.1 and 2.2 one can see that only a limited work about considering fuzzy approaches in the framework of cell-like P systems, but a broad variety of fuzzy cell-like P systems can be further investigated.

- For simplicity, only symport/antiport P systems whose rules only move reactants through membranes were considered to deal with uncertainty in [43], but it is straightforward that this fuzzy approach can be extended to other variants of cell-like P systems.

- Fuzzy mathematics have been used to handle the uncertainty in the number of copies of the reactants, imperfectness of objects in membranes and approximate copies of reactants involved in reactions. These fuzzy mathematics now involve t-norm approach (Triangular norms), fuzzy sets, (general) multi-fuzzy sets, L-multi fuzzy sets, L-fuzzy hybrid sets and so on. Other fuzzy mathematics can also be considered.

- Feasibility of applying fuzzy cell-like P systems in computational biology was discussed in [44], but until now no real application case appears in research literature. So the real application of fuzzy cell-like P systems in computational biology and other fields is a promising research line.

(2) Introducing fuzzy approach into tissue-like P systems. Uncertainty in an inherent property in all living systems and tissue-like P systems are models inspired by the way the tissue cells lives and functions. So uncertainty also should be considered in tissue-like P systems due to the real behavior of tissue cells.

(3) Fault diagnosis method based on FRSN P systems (FDM-FRSNP, for short) is summarized in this paper. However, up to now, the work focuses on the effectiveness and correctness of FDM-FRSNP and the results of application examples are obtained by manual computation.

-To test the speed, convergence and accuracy of the fuzzy reasoning algorithms of FRSN P systems, and to explore automatical generation of FRSN P systems in fault diagnosis, our future work will simulate them on MATLAB [54], P-Lingua [55] or MeCoSim [56]. Meanwhile, how to verify and realize the parallelism of FRSN P systems and their fuzzy reasoning algorithm on hardware such as FPGA and CUDA is also our further task.

-Valuable research interests refer to extend models, algorithms and application areas. For models and algorithms, one promising topic is to design new variants of SN P systems and their reasoning algorithms according to requirements of different fault diagnosis problems, such as on-line diagnosis, fast fault diagnosis, high-precision diagnosis. Another promising interest is to propose FRSN P systems with learning ability. For application areas, FRSN P systems can be used to more different systems, such as power supply systems for urban rail transit, mechanical fault diagnosis and power systems with new energies.

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PI and Fuzzy Control for P-removal in Wastewater Treatment Plant

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Abstract: Due to the complex and non linear character, wastewater treatment process is difficult to be controlled. The demand for removing the pollutant, especially for nitrogen (N) and phosphorus (P), as well as reducing the cost of wastewater treatment plant is an important research theme recently. Thus, in this paper, the benchmark proposed default control strategy and 10 additional control strategies are applied on the combined biological P and N removal Benchmark Simulation Model No.1 (BSM1-P). In addition, according to the results of applying PI controllers, as usual, we also chose the group with the better performance, as well as the default control strategy, to replace the PI controllers with fuzzy controllers. In this way, it can be seen that in all cases the quality of effluent of the controlled process could be improved in some degree; and the fuzzy controllers get a better phosphorus removal.
Keywords: wastewater treatment plant, PI controllers, fuzzy control, P removal.

1 Introduction

As the human society develops rapidly, the demand for water resources is playing a more and more important role in the civil life and industrial production. Nowadays, stringent legislation for wastewater treatment plants (WWTPs) is currently a top driving force for the development of new treatment technologies and for the optimisation of the existing ones. Meeting stringent concentration requirements for Carbon (C), Nitrogen (N) and Phosphorus (P) discharge with minimal costs has raised the need of a more efficient operation.

However, as a large nonlinear system, WWTPs are subject to complex disturbances, where complicated biological and hydrodynamic phenomena are taking place. Thus it is difficult to achieve the aim to meet the standard of WWTPs effluent quality (EQ) and minimize the operational cost (OC) simultaneously. Many control strategies have been proposed in the literature but their evaluation and comparison are difficult, either practical or based on simulation. Different control algorithms for WWTPs have been introduced over the years. For instance, sufficient nitrification can be maintained by applying a constant aeration flow rate, by controlling the dissolved oxygen (DO) level at a pre-selected set-point or by using a variable DO set-point controller based on ammonia concentration in the last aerobic reactor of the plant [1, 2].

On the other hand, the denitrification process is usually controlled by manipulating the external carbon flow rate or internal recirculation flow rate based on nitrate concentration in the last anoxic reactor or in the last aerobic reactor [3, 4]. Unfortunately, various plant configurations, influent characteristics and evaluation criteria have been used in the assessment of control algorithms. As all of these factors influence the choice of a control strategy, it is difficult to say which control algorithm is the most appropriate with respect to minimal OC and best EQ, and whether the implementation of complex control algorithms is really necessary. This is caused by many reasons: the variability of influent (both the volume and the chemical component of influent), the weather condition, and the complexity of biochemical and physical phenomena, the large range of time constants and the lack of standard evaluation criteria, among others. To deal with this problem, in recent years the benchmark simulation model no.1 (BSM1) [5, 6] has been proposed as a standard platform for comparing different control strategies in the community of wastewater treatment processes. This benchmark model can be used for simulating an effective WWTP to reduce the chemical oxygen demand (COD) of the polluted water as well as to remove the nitrogen (N), both of which are key standards for a WWTP.

In addition to COD and N, another major concern is P. Since P has been identified as the key element responsible for eutrophication in the aquatic environment, reducing P release to the environment is an important issue for protecting water resource. Among those activated sludge systems for P removal, enhanced biological phosphorus removal (EBPR) system was notable since it was introduced [20]. In the EBPR system, the phosphate accumulating organisms (PAOs) are responsible for the active of P removal and are enriched to accumulate large quantities of polyphosphate (poly-P) in their cells. In this way the biological P removal is enhanced. However, the PAOs have a stricter requirement of cyclic anaerobic, anoxic and aerobic conditions than N removal and COD removal, thus the process of P removal is more difficult and complex.

Although the BSM1 modelling tool has been widely used in the WWTP research community, it has a structural limitation that it does not involve the P removal that should be taken into account for achieving a more realistic simulation model. To fill this gap, Gernaey and Jorgensen [7] developed a simulation benchmark which models the combined biological P and N removal suitable for the anaerobic-anoxic-oxic (AAO) processes, which could be regarded as benchmark simulation model no.1 including P removal (BSM1-P). Two PI controllers have been designed and tested for this process and defined as the default control (DC). But since there are many potential combinations of control variables, in this paper we propose 10 PI control strategies to compare the control performance. Among them, 5 strategies are basic ones with no more than three controllers, whereas the other 5 strategies are combinations of those 5 basic ones. Besides the traditional PI controller, fuzzy logic controller was drawing more attention for improving the performance of WWTPs due to its model free and easily understandable character.

Fuzzy control can be regarded as a viable alternative control strategy in comparison with the conventional control in some certain situations, e.g. the control process with nonlinear characters which may lead to difficult mathematical modelling and controller tuning. So in this paper, we also give some examples of applying several fuzzy controllers on the WWTP process, and compare the control performance with these referred PI controllers.

2 Description of BSM1-P

The description could be seen in Fig.1. Resembling the BSM1 model, the BSM1-P has a process layout of seven biological reactors and one settler. As it is showed in the Fig.1, the plant lay-out consists of 7 bio-reactors in series followed by a sedimentation tank. Here, Q_{in} means influent, Q_e means effluent, Q_{int} means nitrate recycle, Q_r means sludge recycle and Q_w means waste sludge. The total volume of the biological tanks is 6749 m^3, the volumes of tanks 1, 2, 3 and 4 are 500 m^3, 750 m^3, 750 m^3 and 750 m^3 respectively, which four of them are fully mixed, but not aerated. Tanks 5, 6 and 7 are fully mixed as well as aerated, and their volumes are 1333 m^3. Aeration of tanks 5, 6 and 7 is achieved by using a maximum K_{La} of 10 h^{-1}, here K_{La} means oxygen transfer coefficient. In the openloop situation, default K_{La}s of tanks 5 and 6 are equally set to 10 h^{-1}, and that of tank 7 is set to 2.5 h^{-1}. Dissolved oxygen (DO or S_{O_2}) saturation in arebic tanks is 8 g(-COD)/m^3. The volume of sedimentation tank is 6000 m^3, with area of 1500 m^2 and a depth of 4 m; the sedimentation tank is feed at the point of 2.2 m above the bottom. Two internal recycles are also included: Q_{intr} from tank 7 to tank 3 at a default flow rate of 300% of the influent flow rate, and Q_r from the underflow of the sedimentation tank to the infflent of tank 1. The default Q_r is equal to the Q_{in}. Besides of Q_r, the underfolw of the sedimentation tank is also devided to waste sludge Q_w, and the default Q_w is 400 m^3/d. More detailed explanation about the configuration of combined N and P removal plant can be seen in [7].

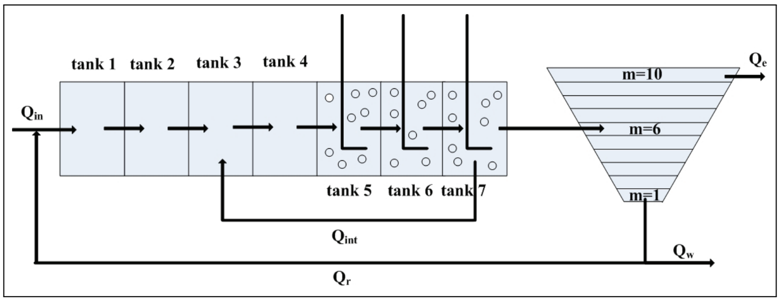

Figure 1: Lay-out of the benchmark plant for evaluation of control strategies on combined N and P removal processes

2.1 Influent composition

The influent for BSM1-P was generated from the ASM1 influent composition [6, 7]. Besides the concentration of pollutant, another important parameter that will affect the operation of WWTP and should be considered is the volumn of influent, which is affected significantly by weather condition. Thus, in BSM1-P, three weather conditions are taken into account: dry weather, rain weather and storm weather.

2.2 Plant performance criteria

It is also necessary to build up a number of indexes to evaluate the performance of the simulated benchmark WWTP studied in this paper. Similar to the original BSM1, the effluent quality index (EQI) for BSM1-P was included P, as calculated by Eqs. (1) and (2).

$$EQI = \frac{1}{1000(t_f - t_0)} \int_{t_0}^{t_f} PU_{(t)} Q_{e(t)} \, dt \tag{1}$$

$$PU_{(t)} = PU_{TSS(t)} + PU_{COD(t)} + PU_{BOD(t)} + PU_{TKN(t)} + PU_{NO3(t)} + PU_{P_{tot}(t)} \tag{2}$$

In Eq. (1), t_0 and t_f represent the start time and end time of the period of evaluating EQI separately. The pollutant load PU_k (kg/d) corresponding to each component k is estimated by Eq. (3).

$$PU_k = \beta_k C_k \tag{3}$$

The factors β_k are weighting factors that are attributed to each effluent component. In this paper, the factors were chosen as follows: $\beta_{TSS} = 2$; $\beta_{COD} = 1$; $\beta_{BOD} = 2$; $\beta_{TKN} = 20$; $\beta_{NO_3} = 20$; $\beta_{P_{tot}} = 20$. Furthermore, the instantaneous concentrations of the different pollutants C_k are calculated by Eqs. (4)-(10).

$$C_{TSS} = X_{TSS} \tag{4}$$

$$C_{COD} = S_F + S_A + S_I + X_I + X_S + X_H + X_{PAO} + X_{PHA} + X_A \tag{5}$$

$$C_{BOD} = 0.25(S_F+S_A+(1-f_{S_I})X_S+(1-f_{XIH})X_H+(1-f_{XIP})(X_{PAO}+X_{PHA})+(1-f_{XIA})X_A) \tag{6}$$

$$C_{TKN} = S_{NH_4} + i_{N,SF}S_F + i_{N,SI}S_I + i_{N,XI}X_I + i_{N,XS}X_S + i_{N,BM}(X_H + X_{PAO} + X_A) \tag{7}$$

$$C_{NO_3} = S_{NO_3} \tag{8}$$

$$C_{N_{tot}} = C_{TKN} + CNO_3 \tag{9}$$

$$C_{P_{tot}} = S_{PO_4}+i_{P,SF}S_F+i_{P,SI}S_I+i_{P,XI}X_I+i_{P,XS}X_S+i_{P,BM}(X_H+X_{PAO}+X_A)+X_{PP}+(1/4.87)X_{MeP} \tag{10}$$

Here, f_{S_I} means fraction of S_I from hydrolysis, $f_{XIA}, f_{XIH}, f_{XIP}$ represent fraction of inert COD from X_A, X_H and X_{PAO}, respectively. In addition, $i_{N,k}$ and $i_{P,k}$ represent N and P fraction in organic component k ($k = S_F, S_I, X_I, X_S, X_H, X_A$ or X_{PAO}), respectively. The influent quality index (IQ) is calculated in the same way as EQI, but the BOD coefficient in Eq. (6) is modified from 0.25 to 0.65.

Similar to BSM1 the limits for certain components should be provided to evaluate the performance of WWTP in detail. By comparing the simulation output with these limits, we could calculate the number of times that the effluent concentration of a pollutant exceeded the limit during the evaluation period. The limit for P is based on the Danish WWTP effluent standard [7], whereas other limits are same to the BSM1, i.e., $C_{P_{tot}}$=1.5 g P/m^3, $C_{N_{tot}}$=18 g N/m^3, C_{BOD}=10 g/m^3, C_{COD}=100 g COD/m^3, C_{TSS}=30 g/m^3 and S_{NH_4}=4 g N/m^3.

In addition, to quantify the cost of WWTP operation, the operating cost index (OCI) was introduced [21]:

$$OCI = \alpha_{EQI}EQI + \alpha_{AE}AE + \alpha_{PE}PE + \alpha_{sldg}P_{sldg} \tag{11}$$

In Eq. (11), EQI is the effluent quality index caculated by Eq. (1), AE is aeration energy consumption rate which happens in aerobic tanks 5, 6 and 7, PE is pumping energy consumption rate to maintain wastewater flowing. The unit for AE and PE is kWh/d. P_{sldg} is the sludge production rate (kg/d). Values for AE, PE and P_{sldg} are calculated in a similar way to BSM1 [6]. The α_i coefficents are OCI weighting factors. In this paper, α_i values are suggested in [21], i.e. α_{EQ}=50 (Euro/year)/EQI; $\alpha_{AE} = \alpha_{PE}$=25 (Euro/year)/(kWh/d); α_{sldg}=75 (Euro/year)/(kg TSS/d).

3 Fuzzy Logic Control

Fuzzy control makes use of so-called fuzzy controllers (FCs) or fuzzy logic controllers to ensure a nonlinear input-output static configuration can be designed/changed according to designer's mind. Compared to the conventional control, fuzzy control could take sufficient advantage of the experience of a human operator, because fuzzy control has the ability to introduce this experience in a more accurate way by applying linguistic variables. The mathematical foundation of fuzzy logic control was set by Zadeh in his paper about forty years ago [8]. After that, as the computer science and the tools for dealing with mathematical problems were developing rapidly, Madamni and Assilian applied the first fuzzy control application on a small steam engine [9,10]. Afterwards, in Japan and USA, and later, in Europe, the fuzzy logic control became more and more popular [19]. Until now, fuzzy controllers have been successfully used in the area of process industries [11–17]. This control method based on human's experience is achieved in FCs by expressing the control requirements and expounding the control signal in terms of the IF-THEN linguistic rules which belong to the set of rules:

$$IF(\textbf{conditions})THEN(\textbf{consequent})$$

Where the **conditions** means the present situation of the controlled process dynamics (compared usually with the desired dynamics), and the **consequent** (conclusion) refers to the action which should be taken - under the form of the control input u - in order to follow the desired dynamics. The set of rules makes up the rule base of the FC.

A typical fuzzy control system is as followed (Fig. 2):

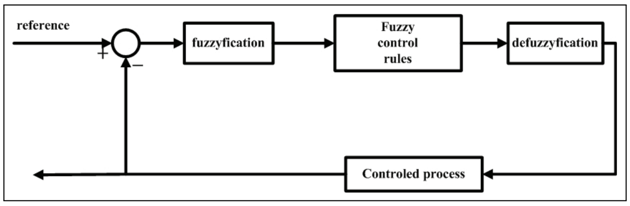

Figure 2: Typical fuzzy control system

In the present study, different forms of fuzzy logic systems for designing FC have been implemented: Mamdani fuzzy inference systems and Sugeno fuzzy inference systems. In a Mamdani fuzzy inference systems [18], given fuzzy rules: (1) if **X** is **A1** and **Y** is **B1** then **Z** is **C1**, (2) if **X** is **A2** and **Y** is **B2** then **Z** is **C2**; and the fact: **X** is **X1**, **Y** is **Y1**, here **X1** and **Y1** are crisp inputs. Fig. 3 shows how to determine the fuzzy output (dark aera). Different from the conventional mathematical set, which only has two relationships with a certain element (*belong to* or *not belong to*), a fuzzy set could also be *partly belonged to* by an element. In the theory of fuzzy, to describe the relationship between a fuzzy element and a fuzzy set, a grade of membership $\mu(x)$ is introduced, and $\mu(x) = 1$ means that the element x totally belongs to a fuzzy set, while $\mu(x) = 0$ means the element x not belongs to the fuzzy set at all. For example, in the Fig. 3, **X1** partly belongs to set **A1** and partly belongs set **A2**, and the grade membership is described by $\mu(\mathbf{X1}|\mathbf{A1})$ and $\mu(\mathbf{X1}|\mathbf{A2})$ separately. Similarly, $\mu(\mathbf{X1}|\mathbf{A1})$ and $\mu(\mathbf{Y2}|\mathbf{B2})$ represent the grade of membership for **Y1** to fuzzy set **B1** and **B2**. This is how to convert crisp inputs to fuzzy inputs, i.e. fuzzification.

Next, we should consider how to determine the conclusions. To do this, we will first consider the recommendations of each given fuzzy rule independently. The membership function for the conclusion reached by rule (1), which we denote by $\mu_{(1)}$, is showed in Fig. 3 and is given by

$$\mu_{(1)}(\mathbf{Z}) = min\{\mu(\mathbf{X1}|\mathbf{A1}), \mu(\mathbf{X1}|\mathbf{A1})\}$$

This membership function $\mu_{(1)}(\mathbf{Z})$ can be explained as how big part we should take into consider for the fuzzy set **C1**. And similarly, we can reach the other membership function for the conclusion by rule (2). And as the present situation (**X** is **X1** and **Y** is **Y1**) is affected by both rule (1) and rule (2), thus the final decision should be a combination of both membership funcion, as showed in Fig.3.

According to centroid way, to defuzzify the fuzzy output, we only need to calculate the centroid point of the gray aera: let b_i denote the center of the membership function of the sonsequent of rule (i), and let $\int \mu(i)$ denote the area under the membership function $\mu_{(i)}$, the centroid method computes **Z1** to be

$$\mathbf{Z1} = \frac{\sum_i b_i \int \mu(i)}{\sum_i \int \mu(i)}$$

and **Z1** is the defuzzified output, as showed in Fig. 3.

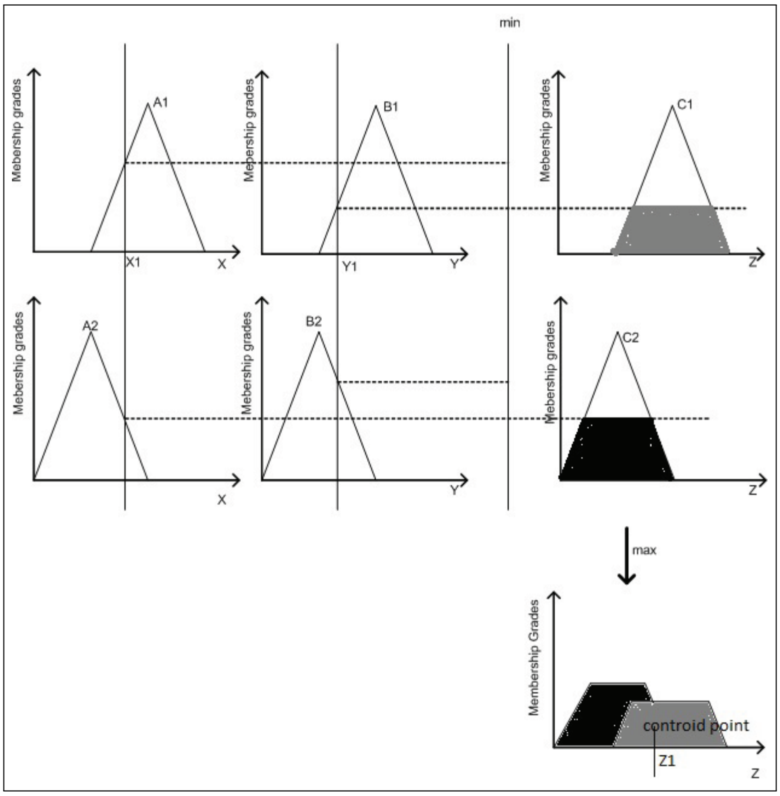

Figure 3: Mamdani fuzzy inference systems

To design a fuzzy controller, at first the input and output variables of the fuzzy controller should be chosen. Normally the input variables are feedback error and higher order derivatives of feedback error, in this paper the feedback error (E) and first order derivative of feedback error (EC) were chosen as inputs of the fuzzy controller. On the other hand the output variables should be controlled inputs of the controlled plant (U), therefore in this case, the K_{La} of the aerobic tanks was chosen as output of the DO fuzzy controller, and the internal recycle rate was chosen as the output of the internal recycle fuzzy controller.

The next step is to choose the membership functions for input and output variable. The shape of membership functions here we chose the triangle functions both for the input variables and for the output variables. It was concluded that in the case of tank 5 the range of E could be fixed from -1.5 g/m^3 to 1.5 g/m^3, the range of EC could be fixed from -15g/ (m^{3}d^{-1}) to 15 g/ (m^{3}d^{-1}). The K_{La} of tank 5 could range from 160 d^{-1} to 280 d^{-1}. In addition, the number of parameters of the membership function was chosen as 7, which included NB, NM, NS, O, PS, PM, PB. Here N meant negative, O meant zero, P meant positive, B meant big, M meant medium, S meant small. To express in a more clear way, Fig.4 presents the membership function of the feedback error for the fuzzy controller of the tank 5.

Similarly, this could also be applied on the DO fuzzy controller of tank 6, tank 7 and that of the internal recycle. But in the case of tank 6 and 7, E and EC range the same way, because

Table 1: Decision table for fuzzy controllers

(a) For DO fuzzy controllers

	NB	NM	NS	O	PS	PM	PB
NB	PB	PB	PB	PB	PM	O	O
NM	PB	PB	PB	PB	PM	O	O
NS	PM	PM	PM	PM	O	NS	NS
O	PM	PM	PS	O	NS	NM	NM
PS	PS	PS	O	NM	NM	NM	NM
PM	O	O	NM	NB	NB	NB	NB
PB	O	O	NM	NB	NB	NB	NB

(b) For internal recycle

	NB	NS	O	PS	PB
NB	PB	PB	PB	PS	O
NS	PS	PS	PS	O	NS
O	PS	PS	O	NS	NS
PS	PS	O	NS	NS	NS
PB	NS	NB	NB	NB	NB

according the experiments the range of E and EC did not affect the control performance significantly, and the U ranged from 120 d^{-1} to 240 d^{-1} and from 60 d^{-1} to 180 d^{-1} separately. In the case of internal recycle, the E ranged from -1 g/m^3 to 1 g/m^3, the EC ranged from -20 g/ (m^3d^{-1}) to 20 g/ (m^3d^{-1}) and the U ranged from 5000 m^3/d to 45000 m^3/d. The number of parameters for U of the internal recycle is adjusted to 5, which means that only exist NB, NS, O, PS and PB.

The next step was deciding the fuzzy inference mechanism. In the case of 7 parameters of linguistic terms, there are totally 49 control rules formed by if-then clauses. Table.1(a) shows the detail of control rules, which are same for the entire 3 DO fuzzy controller, whereas Table.1(b) shows the fuzzy control rules for the internal controller.

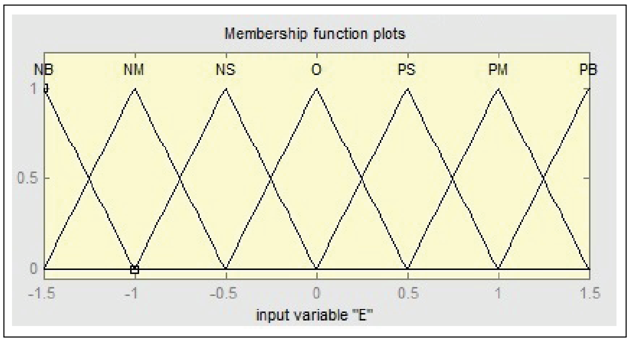

Figure 4: Membership of E of tank 5

4 Control Configurations

In this paper, at the first part, a series of control strategies are applied on the BSM1-P. As given in [7], a default control (**DC**) is simulated as a reference as well as a test for the updated simulation plant. Then 10 additional PI-based control strategies (**S1-S10**) are applied to compare the performance: the first 5 control strategies (**S1-S5**) are basic ones, including the DO controller, cascade DO controller, internal recycle flow rate controller, extra carbon resource controller and the waste sludge amount controller, whereas the other 5 control strategies (**S6-S10**) are generated by combining these basic ones. A detailed description of each control strategy follows below.

4.1 Default Control (DC) Strategy

Similar as with the original BSM1, the **DC** strategy consists of two PI-based control loops: a DO controller in tank 7 and an internal recycle controller. The measured variables are dissolved oxygen of the tank 7 and the concentration of nitrate nitrogen (NO_3) of the tank 3 respectively. The controlled variables are same as measured variables, and the set points are DO=2 g/m^3 and NO_3=1 g/m^3 respectively. The manipulated variables are K_{La7} and internal recycle Q_{int} respectively. As showed in Fig. 5.

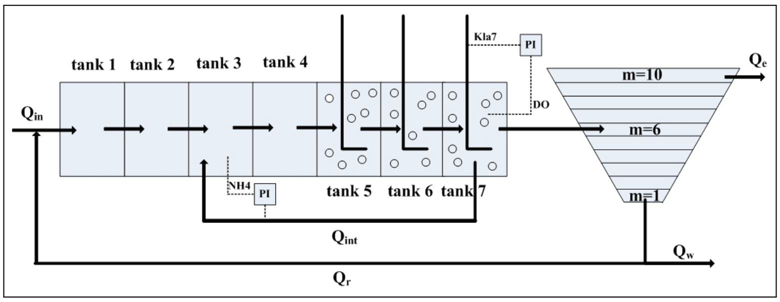

Figure 5: Configuration of DC

4.2 Control strategies configurations

A set of 5 basic control strategies (**S1** to **S5**) has also been implemented by using the following control loops respectively:

- **S1**: PI Controllers of the dissolved oxygen concentration (DO) in the 3 aerobic tanks by regulating the oxygen transfer coefficients (K_{La}) simultaneously, and the set points are all 2 g/m^3.Internal recycle loop is left as in the openloop. The configuration is shown in Fig. 6.

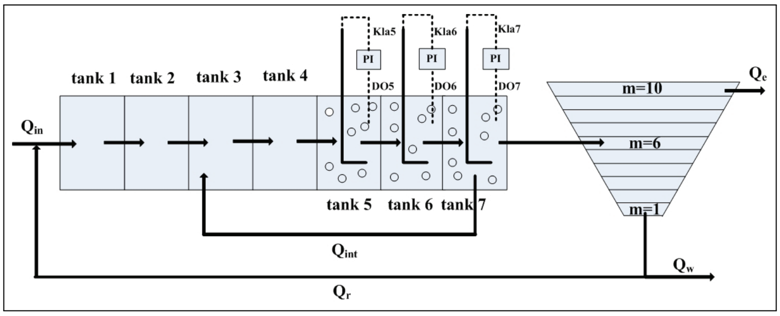

Figure 6: Configuration of S1

- **S2**: Fig.7 shows the cascade PI control of the ammonia nitrogen of the effluent by manipulating the DO set points in all the aerobic tanks. The set point of effluent ammonia nitrogen concentration is 1 g/m^3, and the controlled variables are also the K_{La}s of the aerobic tanks. Internal recycle loop is left as in the openloop.

- **S3**: Fig.8 shows the control of nitrate nitrogen concentration (S_{NO3}) in the tank 4 by manipulating the internal recycle flow rate (Q_{int}). The set point is 1 g/m^3. And the K_{La}s of aerobic tanks are left constant as in the openloop.

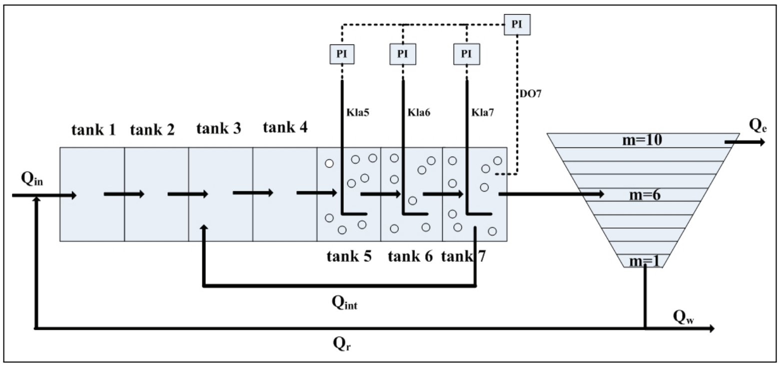

Figure 7: Configuration of S2

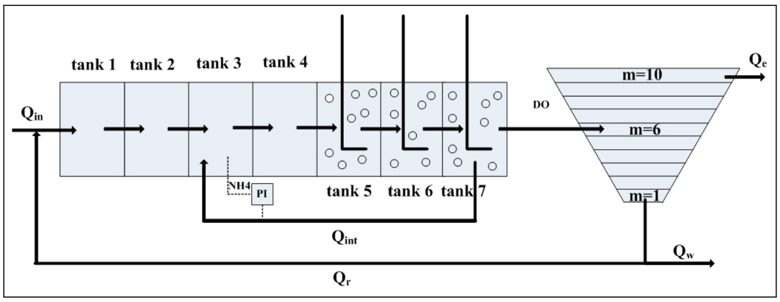

Figure 8: Configuration of S3

- **S4**: Fig.9 shows the control of S_{NO3} in the tank 4 by manipulating the extra addition carbon resource (Q_{carb}) into the tank 3. The set point is also 1 g/m^3. And the internal recycle loop was left as openloop.

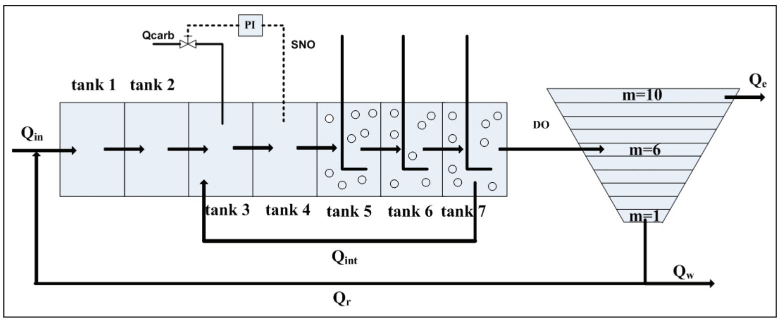

Figure 9: Configuration of S4

- **S5**:Fig.10 shows the control of total suspended solids concentration (X_{TSS}) in tank 7 by manipulating the wastage sludge flow rate (Q_w). The set point is 4000 g/m^3. And the internal recycle loop was left as openloop.

Furthermore, 5 extra PI-based control strategies (**S6** to **S10**) generated by combining these basic control strategies are also tested. In detail, the control strategy **S6** is to control the DOs in the 3 aerobic tanks as well as internal recycle flow rate by applying **S1** and **S3** simultaneously. As can be seen in the simulation result, the performance of **S2** is not beneficial for the phosphorus removal, which is mainly considered in this work, consequently in all the combined control strategies, none is included **S2**. The control strategy **S7** is obtained by combining **S1** and **S4**,

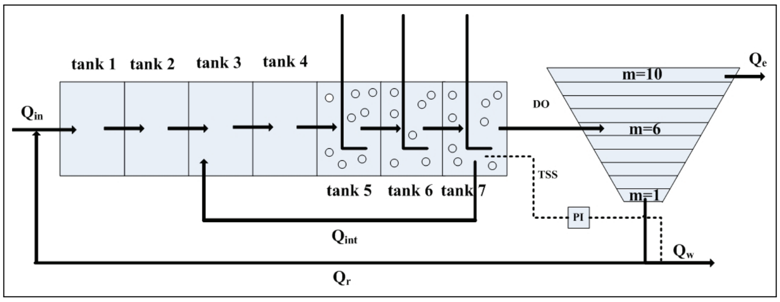

Figure 10: Configuration of S5

which means control the DO in 3 aerobic tanks and the extra carbon resource in tank 3. Similarly, the control strategy **S8** is to combine **S1** and **S5**, **S9** is to combine 3 control strategies (**S1**, **S4** and **S5**). Finally, the last control strategy **S10** is generated by combining **S1**, **S3** and **S5**.

5 Results and Discussions

The important information of simulation results of PI-based control strategies is showed in the Table.2. For comparison, it included the simulation result of open loop, default control loop and **S1** to **S10** control strategies. The effluent quality indexes (EQIs) are showed in the table to judge the overall performance. In detail, the mainly considered components of a certain wastewater treatment plant: ammonia nitrogen (NH_4), total amount of nitrogen (N_{tot}), phosphate (S_{PO_4}), the amount of chemical oxygen demand (COD) and the total suspended solids (TSS) being another main factor to evaluate the performance of a WWTP are showed in the table. In addition, the operation cost index (OCI), as a consequence of consuming aeration energy (AE), pumping energy (PE), sludge production and the added carbon volume and metal volume (in this paper, 0 in all case), is also given in the table to compare the control strategies.

Table 2: Results of PI-based control strategies (**S1-S5**)

Average effluent Concentration		default	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10
component (g/m3)	limit											
SNH4	4	3.16	3.36	**2.17**	2.84	3.62	2.30	3.36	4.02	2.61	3.68	2.60
Ntot	18	17.15	15.19	15.06	15.47	**12.59**	15.16	15.36	12.91	14.82	12.66	15.01
SPO4	2	1.86	2.55	3.99	3.150	0.85	2.97	2.27	0.85	2.19	**0.82**	1.97
COD	125	**45.44**	45.46	45.58	45.52	45.90	46.17	45.45	45.79	46.04	46.05	46.02
TSS	35	14.15	14.05	**13.93**	14.01	14.54	14.58	14.08	14.47	14.59	14.66	14.60
Global plant Performance												
EQI(kg/d		4495	4741	5618	5468	4502	5403	4643	**4087**	4672	4096	4597
OCI(euro/d)		19919	19664	19779	19343	24858	**18954**	19774	24231	19532	24373	19648

Since BSM1-P is an updated version of BSM1, from the results it can be seen that in all cases the ammonia nitrogen (NH_4), total amount of nitrogen (N_{tot}) and the chemical oxygen demand (COD) are all under limit amount.

In case of **S4** and those that included it: **S7** and **S9**, all the limit concentrations for pollutant components concerned are satisfied. The reason is that extra carbon resource can promote the growth and ability of denitrifying organism and phosphorus accumulating organism (PAO) simultaneously. However because of the added carbon resource, the operational cost indexes

(OCI) in these 3 cases are higher than any other case.

In the other cases of PI-based control strategies, it is obvious that, by comparing with the case of openloop, a lower level of ammonia nitrogen concentration in the effluent corresponds a higher level of phosphorus concentration. This is because without extra carbon resource, the only way to decrease the ammonia is to increase the explosion of air, but this will lead a higher level of nitrate nitrogen, which is harmful for the accumulation of phosphorus.

However, the case **S5** is an exception, where both S_{NH_4} and S_{PO_4} are lower than the case of openloop. Although the phosphorus is still above the limitation, but considering the concentration of ammonia nitrogen is relatively lower than many cases, the consequence is also beneficial. This is because by controlling the waste sludge flow rate more suspended solids, including the organisms, remain in the treatment plant circumstance, which is beneficial for both nitrogen and phosphorus removal. Correspondingly, the COD and TSS in the effluent are higher than other cases. Furthermore, the OCI is the lowest.

Fig.11 shows the tradeoff among OCI and EQI for the different PI-based control strategies. In Fig.11, the horizontal axis means the operational cost index (OCI) and the vertical axis represents the effluent quality index (EQI). Basically, higher operational cost leads to lower effluent quality, which means that the performance of treatment plant is better. From Fig.11, it is easy to conclude that those strategies including extra carbon flow (**S4**, **S7**, **S9**) cost much more than other ones, but achieve lower effluent pollutant concentration. In fact, the performance of **S4** is the best among the 5 basic PI-based control strategies. Among all, **S7** and **S9** possess the lowest effluent pollutant load without significant difference. However, when analyzing the control strategies combined with the operational cost, the default control, **S1**, **S6**, **S8** and **S10** show a good balance. But the main flaws of them are as followed: the average level of phosphorus in **S6** and **S8** exceeded the limit amount, and the instant level of phosphorus amount in all the 5 cases violated the limitation for a great partial of the total evaluated time (64.43%, 36.01%, 49.55%, 46.88% and 38.69%, respectively).

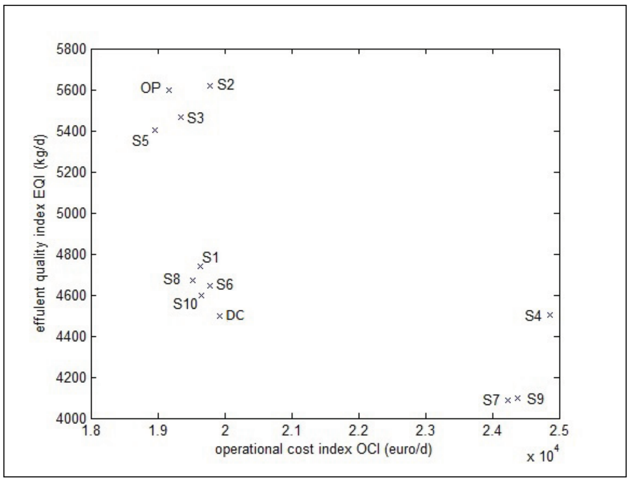

Figure 11: The OCI against EQI graph of PI controllers

In Fig.11, the points that are closer to the origin mean lower effluent quality index with less operational cost, therefor better tradeoff, when choosing the appropriate control, these ones should receive more interest. Hence it is obvious that **S10** made the best balance enter OCI and EQI. Fig.12 and Fig.13 show a dynamic plant performance of WWTP for N and P under the PI-based control strategy **S10**. To make a clear comparison with the fuzzy controller, in these three figures we also added in the time response of certain components of effluent by using fuzzy controllers, which will be anylized below.

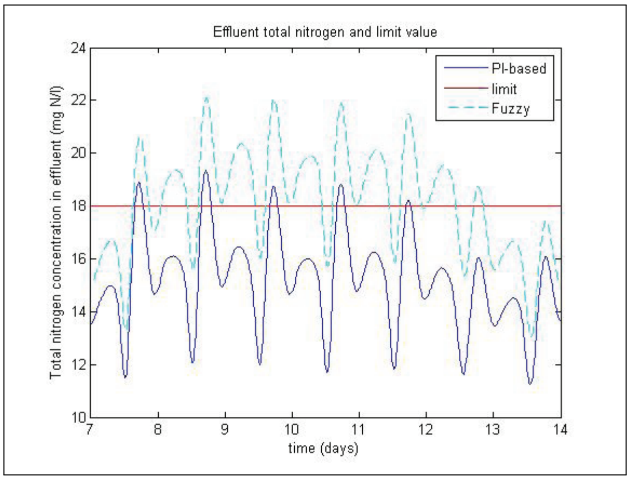

Figure 12: Total amount of nitrogen of effluent of S10

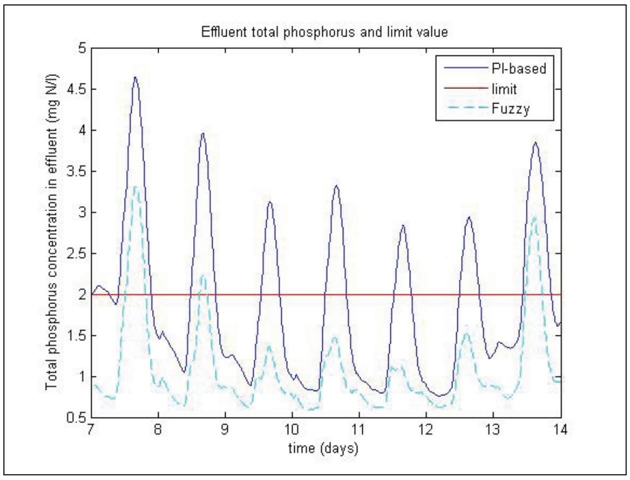

Figure 13: Total amount of phosphorus of effluent of S10

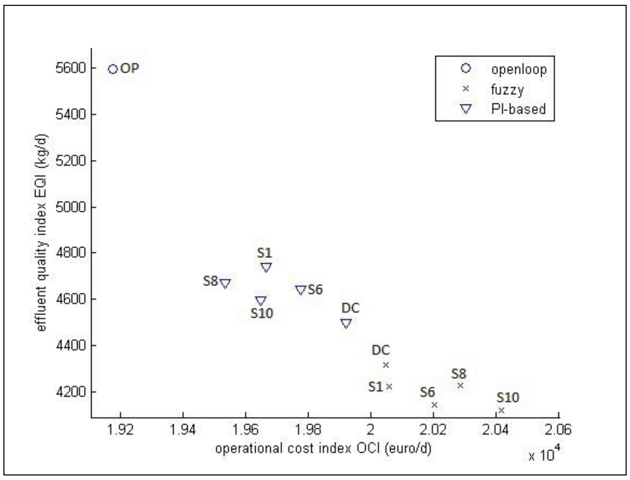

Figure 14: The comparison of OCI to EQI between PI-based and Fuzzy-based controllers

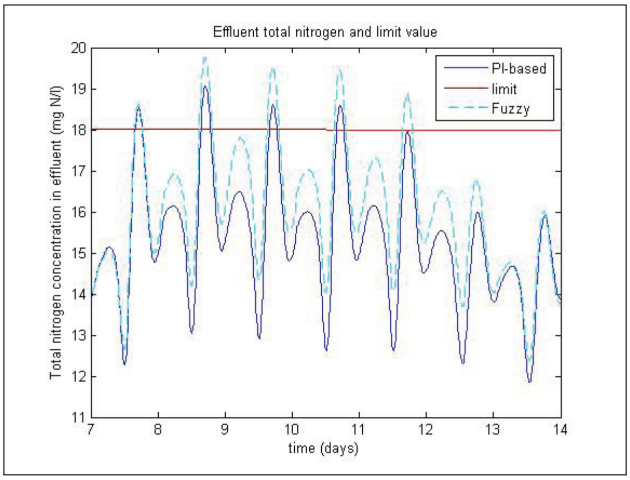

Figure 15: Total amount of nitrogen of effluent of S1

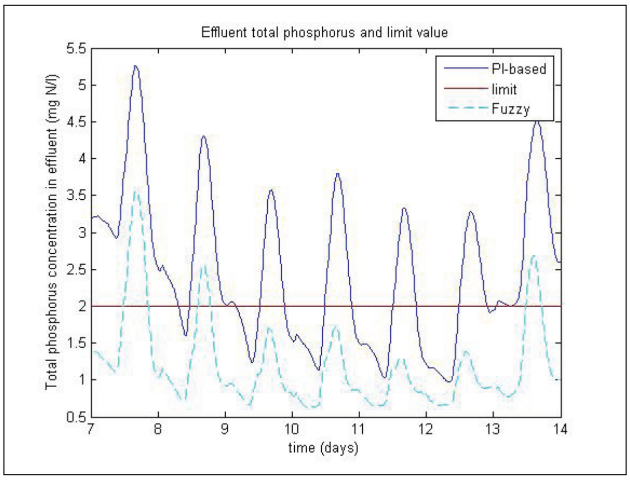

Figure 16: Total amount of phosphorus of effluent of S1

Table 3: Comparison of PI and Fuzzy control strategies

average effluent concentration		open loop	default		S1		S6		S8		S10	
			PI	FUZZY	PI	FUZZY	PI	FUZZY	PI	FUZZY	PI	FUZZY
component (g/m3)	limit											
SNH4	4	2.79	3.16	5.77	4.43	6.99	3.36	6.54	2.61	6.55	**2.60**	7.11
Ntot	18	15.50	17.15	17.04	15.19	15.91	15.36	17.73	**14.82**	16.42	15.01	18.10
SPO4	2	3.69	1.86	1.52	2.55	1.27	2.27	**1.08**	2.19	1.46	1.97	1.15
COD	125	45.54	45.44	45.36	45.46	45.40	45.45	45.32	46.04	44.80	46.02	**44.83**
TSS	35	13.95	14.15	14.12	14.05	14.17	14.08	14.12	14.59	**13.62**	14.60	13.69
EQI(kg/d)		5596	4496	4314	4741	4224	4643	4141	4672	4226	4597	**4121**
% of variation		-	-19.7%	-22.9%	-15.3%	-24.5%	-17.0%	-26%	-16.5%	-24.5%	-17.9%	**-26.4%**
OCI(euro/d)		19175	19920	20047	19665	20057	19774	20203	**19532**	20283	19648	20416
% of variation		-	+3.9%	+4.5%	+2.6%	+4.6%	+3.1%	+5.4%	**+1.9%**	+5.8%	+2.5%	+6.5%

Fig.12 and Fig.13 show the most concerned component of waste water, nitragen (N) and phosphorus (P) separately. From Fig.12, we can see that although the average amount of Ntot is satisfied with the requirement according to Table.2, in some moments the Ntot of effluent violates the limitation. However, the overall performance is rather good. When come to Fig.13, the outcome is in contrast, the average amount of P of effluent is higher than the desired limit, and neither is there much time satisfying the limitation. So this is the major defect of **S10**. But considering the overall control performance and the cost of WWTP, **S10** still draws a good attention. So in the next step, we focus on replacing the PI controllers of **S10** by fuzzy controllers. In addition, since DF, **S1**, **S6** and **S8** also get a good comparison of OCI and EQI, as well as a good ability for P removal, as can be seen in the Table.2 and Table.3, it is also necessary to build up a fuzzy-based controller for these control strategies to see the performance. The procedure and important information of designing fuzzy controller are mentioned in the second part.

The simulation results of fuzzy control strategy are showed in Table.3, to make a clear comparison, in Table.3 we also repeat the situation of applying PI-based controllers. From this table, we can see that by applying FCs, the average concentration of P in effluent becomes much lower and satisfies with the requirement. But since the favorable condition for P removal is contrary against the one of N removal, by applying FCs the amount of N rises. However, the average concentration of total N in effluent is still under the limitation (except fuzzy-based **S10**), this means that FCs are able to satisfy with the requirement of total N and total P simultaneously. In addition, from the table we can also see that by applying FCs, the EQI is lower, but the OCI is higher. To make a clear comparison, Table.3 also gives a percent (%) of variation of each control strategy against openloop to see in how much degree OCI enlarged as well as EQI reduced. From Table.3, we can see that in DC by applying fuzzy controllers EQI reduced about 3% more than by applying PI controllers, and OCI only gained 0.6%. In other control configurations, by applying fuzzy controllers, EQI reduced 8%-9% comparing with PI controllers, and OCI gained about 2%-4%. This means fuzzy logic controllers are able to improve the WWPT performance by increasing operational cost, however the degree of improvement is greater than the increase in cost. Fig.14 gives a more clear way to see this conclusion. To see the dynamic plant perform, we can refer to Fig.12 and Fig.13. It can be concluded that by applying fuzzy controllers in every moment P gets a better removal.

Another phenomenon that is exihibited in Fig.15 and Fig.16 needs to be mentioned, which show effluent concentrations of N and P of WWTP controlled by **S1**.Since **S1** only contains DO controllers, these two pictures can reveal the effect of DO on N and P removal process, because despite of the same setpoint for both PI-based and fuzzy-based controllers, the regulated instantaneous DO could vary according to different type of controllers. From Fig.15, we can see that the PI controllers are beneficial for the N removal, but in some period (such as Day7 to Day8 and Day13 to Day 14) the concentration of N did not change much, but from Fig.16 we can see that all the time the amount of P by using Fuzzy controllers is below the one by using PI controllers. From this point, the EQI of fuzzy controllers is lower than the PI controllers, as showed in Table.3. It can be assumed that the process of P removal is more sensitive than the N removal. Based on this assumption, the real concentration of DO in the fuzzy-based **S1** at the moment that N keeps the same could draw more interest, because under this condition we can get better removal of P and at the same time do not affect the removal of N.

From the simulation results, we can see that by applying Fuzzy control strategies the performance of WWTP is improved, and fuzzy control strategy could get a better operation result for the wastewater treatment process in some degree.

6 Conclusions

In this paper, at first a set of PI control strategies has been applied on the BSM1-P waste water treatment plant to maintain the pollution component of effluent within regulations specified limits. Good performance was achieved; it could be proved that the BSM1-P is efficient to simulate the combined P and N removal WWTP. Among the 10 designed PI control strategies, we chose a group that could make a better balance between OCI and EQI (DC, **S1**,**S6**, **S8** and **S10**) to test fuzzy logic controllers. From the results we could conclude that by applying FCs the P removal consequent was enhanced. In this way, the most focused on component (P) of the effluent was controlled under the required limit. The results show FC could be efficiently used to control the WWTP, especially for the P removal.

However, in this paper, although a set of fuzzy control strategies was tested on the BSM1-P, the operation was only replacing the FCs to the PI controllers. The control loop was not changed at all and the set points were also the same. But as one of the most studied advanced control strategies, fuzzy logic control could make a better improvement. Therefore, the future work will not only concern to replace the FC to the PI controllers, but also make a combination of fuzzy logic control and PI control. FCs could act as a higher level to make the important decision as a human being, and the basic control loop could be accomplished by PI controller. For example, when we focus on the concentration of P of effluent, it is not necessary to fix the DO of three aerobic tanks to 2 mg/l; high level of DO is beneficial for the P removal, but harmful to the denitrification process which is important for the N removal. Although according to the reference, 2 mg/l of DO is an ideal amount for making the balance between P removal and N removal, however in the real situation there are so many disturbances in the WWTP, it would get a better control performance by adjusting the DO according to the specific situation. In this way, we could get a better balance between P removal and N removal, as well as between OCI and EQI.

Acknowledgments

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Book Review: "Biometric and Intelligent Decision Making Support"

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Biometric and Intelligent Decision Making Support
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This book is meant to familiarize the reader with the state-of-art of research and application results obtained in the domain of Intelligent Decision Support Systems (IDSS), biometric technologies and their integration. The author of the book is prof. A. Kaklauskas, the head of the Department of Construction Economics and Real Estate Management of the Vilnius Gediminas Technical University, Faculty of Engineering, Vilnius, Lithuania, a reputed author in the domain of Intelligent Decision Support Systems, Multi-criteria Decision Making (MCDM) methods and their applications in the construction, property management and the related fields. In the book, the author makes an exhaustive review of results reported in the domain literature and presents a detailed account of his practical achievements as well as his colleagues'.

The book is composed of seven chapters as follows.

Chapter 1, entitled "Introduction to Intelligent Decision Support Systems", sets the stage for the following chapters. It introduces the main concepts of the domain and pays a special attention to Artificial Intelligence (IA)- based advanced methods and their integration with Data Base Management Systems (DBMS) and Human- Computer Interface (HCI).

Chapter 2, entitled "Intelligent Decision Support Systems", reviews basic concepts of several new subclasses of IDSS, such as recommender, advisory and expert systems, data and text mining. The integration of Artificial Neural Networks is addressed too. A big part of the chapter provides abundant information about specific aspects of biometrics-based DSS, such as voice and speech recognition, expression analysis and so on. Two very modern concepts and associated technologies, namely Ambient Intelligence and Internet of Things, are described in details. Several other subclasses of solutions based on various technologies, such as Genetic algorithms, Fuzzy and Rough sets, Computer vision, robotic systems and so on, are surveyed in the final part of the text of the chapter.

The remaining part of the text contains the presentation of various conceptual and application results obtained by professor Kaklauskas and his colleagues. Chapter 3 introduces the model of passive house to be used in a qualitative analysis and design of the corresponding intelligent system.

Chapter 4 contains the results obtained in biometric and self-assessment of the student progress system and describes two relevant case studies.

Chapter 5 addresses the subject of analysis of user's emotions and work productivity by using the state of blood circulatory system and describes a web- based biometric computer mouse advisory system.

The content of chapter 4 is complemented by the interesting solutions presented in chapter 6, which addresses student progress assessment by using an intelligent pupil analysis and presents two case studies.

Chapter 7 addresses also the education domain and presents recommender-system-based solutions meant to increase student productivity.

One book addresses two modern research domains: intelligent and integrated decision support systems and biometrics-based human-computer interface. It can be explained, in my view, by the huge effort made by the author and his colleagues to find solutions to problems of the real world. The presentation of the research and application results obtained by the author and his colleagues over the years is a special merit of the book, beside the variety of the topics addressed. Consequently, I warmly recommend the book to PhD and MSc students and instructors, as an up-to-date source of knowledge. I think the book can also be a valuable tool for those consultants who are willing to master new concepts and technologies in order to get a competitive advantage on the market of computer applications.

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Author index