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From Function Approximation to Swarm Dynamics

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Abstract

We show that an interesting and important problem of design of dynamics for mobile agents in a swarm is nicely addressed via mean-square-integrated-loss optimization for function approximation in a context of mixture modeling.

Keywords: Function approximation, swarm dynamics, least squares optimization, multi-agent systems, gradient descent.

1 Introduction

Assume we wish to deploy a swarm of agents on the plane in order to carry out a global task that involves dispersing so as to satisfy a "demand" function $\Psi(x,y) \geq 0$, weighting the importance to "cover" the location (x,y) on the plane. Every agent in the swarm, when located at $P_i \triangleq (x_i, y_i)$, provides "cover" of its neighborhood according to a sensing and influence profile given by $\varphi_i(x-x_i, y-y_i) \geq 0$.

If N agents are available, and they are located at points

$$\left\{P_1 \triangleq (x_1,y_1), P_2 \triangleq (x_2,y_2), \ldots, P_N \triangleq (x_N,y_N)\right\}, \tag{1}$$

their "joint coverage" of the plane is taken to be

$$\mathcal{R}_{\{P_1,P_2,\ldots,P_N\}} = \sum_{i=1}^{N} \varphi_i(x-x_i, y-y_i). \tag{2}$$

We need the agents to move in order to form a spatial constellation that best satisfies the demand, i.e. we need to minimize the "coverage vs. demand" difference function

$$\mathcal{E}(x,y) \triangleq \mathcal{R}_{\{P_1,P_2,\ldots,P_N\}} - \Psi(x,y). \tag{3}$$

Ideally, we would like to have that the coverage of every point is at least $\Psi(x,y)$, hence we want

$$\mathcal{R}_{\{P_1,P_2,\ldots,P_N\}} = \sum_{i=1}^{N} \varphi_i(x-x_i, y-y_i) \geq \Psi(x,y) \tag{4}$$

for all $(x,y) \in \mathbb{R}^2$, implying that we need to have $\mathcal{E}(x,y) \geq 0$ everywhere. Recall that we assume the demand $\Psi(x,y)$ is nonnegative everywhere and also that $\varphi_i(x,y)$ are positive coverage or "sensing" functions. The coverage condition $\mathcal{E}(x,y) \geq 0$ can surely be achieved if we have a finite demand region with bounded $\Psi(x,y)$ and we have a huge number of agents each contributing some significant "sensing" to the joint sensing profile of the configuration. However, this is rarely the case. While $\Psi(x,y)$ is usually bounded and lives on a finite region in the plane, it is usually the case that we have a limited supply of mobile agents, and their integrated sensing function $\mathcal{R}_{\{P_1,P_2,...,P_N\}}$ can barely meet or approximate the demand. We therefore consider an integrated demand satisfaction error of the form

$$\Phi_{\{P_1,P_2,...,P_N\}} = \iint\limits_{\Omega} \mu_{(x,y)} \left(\mathcal{R}_{\{P_1,P_2,...,P_N\}} - \Psi(x,y) \right) dxdy \tag{5}$$

where $\mu_{(x,y)}(\cdot)$ is a function penalizing the local error $\mathcal{E}(x,y)$, which can be either uniform over the region of interest Ω (usually the region where $\Psi(x,y)$ is positive) or it may also depend on the "importance" of location (x,y). A natural choice for $\mu_{(x,y)}(\cdot)$ is the squared error measure, yielding

$$\Phi_{\{P_1,P_2,...,P_N\}} = \iint\limits_{\Omega} \left(\mathcal{R}_{\{P_1,P_2,...,P_N\}} - \Psi(x,y) \right)^2 m(x,y)\, dxdy \tag{6}$$

i.e. $\mu_{(x,y)}(\cdot) = m(x,y)(\cdot)^2$, $m(x,y)$ being given as a local "importance" function. We note that usually $m(x,y)$ is constant over the region of interest.

Other choices for $\mu_{(x,y)}(\cdot)$ are possible. The (integrated) squared error functional equally punishes "under" and "over" coverage of locations (x,y), i.e. positive error of over-coverage is as bad as lacking coverage levels. One might want to penalize under-coverage more severely than over-cover, however when the number of agents is limited, a symmetric penalty is quite reasonable since we want to achieve adequate coverage by efficiently exploiting the *sensing capabilities* of the agents in the swarm.

2 Least Squares Design

We have discussed a problem of multi-agent coverage to meet a given demand function $\Psi(x,y)$ and this led us to set up a "demand satisfaction error" minimization problem of the form

$$\min_{\{P_1,P_2,...,P_N\}} \iint\limits_{\Omega} \left(\Psi(x,y) - \sum_{i=1}^{N} \varphi_i((x,y) - P_i) \right)^2 m(x,y)\, dxdy. \tag{7}$$

This problem is familiar from the field of signal approximation, or representation, where it usually takes the form of "describing" functions of interest with the help of a given standard sequence of "basic" functions, as weighted superposition as follows:

$$\Psi(x,y) \sim \tilde{\Psi}(x,y) = \sum_{i=1}^{N} \gamma_i \beta_i(x,y). \tag{8}$$

The set of "standard" functions $\{\beta_i(x,y)\}_{i\in\mathbb{N}}$ often constitute an orthonormal sequence of "basis" functions, making the computation of the weights γ_i very easy. In our case the agents have a "coverage" sensing function of the form $\varphi_i(x-x_i, y-y_i)$, hence the functions we wish to approximate $\Psi(x,y)$ with are "shifted" coverage profiles. Assuming all agents in the swarm are identical, we have that $\varphi_i(x-x_i, y-y_i) = \varphi(x-x_i, y-y_i)$. Hence our approximation / representation problem becomes

$$\Psi(x,y) \rightsquigarrow \tilde{\Psi}(x,y) = \mathcal{R}_{\{P_1,P_2,...,P_N\}} = \sum_{i=1}^{N} \varphi((x,y) - P_i). \tag{9}$$

Note that instead of having weighting coefficients, we have here *spatial location shifts* describing the agent's spatial constellation or configuration.

While solving for the representation of $\Psi(x,y)$ with a set of weighted "basis" functions $\{\beta_i(x,y)\}_{i\in\mathbb{N}}$, the weights can be computed by solving a linear set of equations, the problem of solving for the shifts

P_i in our case is more complex and interesting. We shall try to address this optimization by a gradient descent type approach, and this leads to a dynamical system moving the points P_i from some initial constellation to an optimal constellation, i.e. to a stationary point of the optimization functional.

Consider the minimization problem

$$\min_{\{P_1,P_2,...,P_N\}} \mathcal{F}_{\{P_1,P_2,...,P_N\}}, \tag{10}$$

$$\mathcal{F}_{\{P_1,P_2,...,P_N\}} = \iint\limits_{\Omega} \left(\Psi(x,y) - \sum_{i=1}^{N} \varphi(x-x_i, y-y_i) \right)^2 m(x,y)\, dxdy,$$

and compute the gradient of $\mathcal{F}$ w.r.t. the variables $\{P_k = (x_k, y_k)\}$, $k = 1, 2, \ldots, N$. We have:

$$\frac{\partial \mathcal{F}}{\partial P_k} = \begin{bmatrix} \dfrac{\partial \mathcal{F}}{\partial x_k} \\ \\ \dfrac{\partial \mathcal{F}}{\partial y_k} \end{bmatrix} \tag{11}$$

$$= -2 \iint\limits_{\Omega} \left(\Psi(x,y) - \sum_{i=1}^{N} \varphi(x-x_i, y-y_i) \right) \begin{bmatrix} \frac{\partial \varphi}{\partial x}(x-x_k, y-y_k) \\ \\ \frac{\partial \varphi}{\partial y}(x-x_k, y-y_k) \end{bmatrix} m(x,y)\, dxdy$$

$$= \begin{bmatrix} -2 \iint\limits_{\Omega} m(x,y)\, \Psi(x,y)\, \frac{\partial \varphi}{\partial x}(x-x_k, y-y_k)\, dxdy \\ +2 \sum_{i=1}^{N} \iint\limits_{\Omega} \varphi(x-x_i, y-y_i)\, \frac{\partial \varphi}{\partial x}(x-x_k, y-y_k)\, m(x,y)\, dxdy \\ \\ -2 \iint\limits_{\Omega} m(x,y)\, \Psi(x,y)\, \frac{\partial \varphi}{\partial y}(x-x_k, y-y_k)\, dxdy \\ +2 \sum_{i=1}^{N} \iint\limits_{\Omega} \varphi(x-x_i, y-y_i)\, \frac{\partial \varphi}{\partial y}(x-x_k, y-y_k)\, m(x,y)\, dxdy \end{bmatrix};$$

hence,

$$\frac{\partial \mathcal{F}}{\partial x_k} = -2 \iint\limits_{\Omega} \Psi(x,y)\, \frac{\partial \varphi}{\partial x}(x-x_k, y-y_k)\, m(x,y)\, dxdy \tag{12}$$

$$+2 \sum_{i=1}^{N} \iint\limits_{\Omega} \varphi(x-x_i, y-y_i)\, \frac{\partial \varphi}{\partial x}(x-x_k, y-y_k)\, m(x,y)\, dxdy,$$

i.e. $\dfrac{\partial \mathcal{F}}{\partial x_k}$ is composed of a factor determined by the "demand" in interaction with the coverage profile of the agent at P_k and N terms determined by the interaction of the coverage profile of the agent at P_k with the coverage profiles of all other agents (and itself too!). Similarly,

$$\frac{\partial \mathcal{F}}{\partial y_k} = -2 \iint\limits_{\Omega} \Psi(x,y)\, \frac{\partial \varphi}{\partial y}(x-x_k, y-y_k)\, m(x,y)\, dxdy \tag{13}$$

$$+2 \sum_{i=1}^{N} \iint\limits_{\Omega} \varphi(x-x_i, y-y_i)\, \frac{\partial \varphi}{\partial y}(x-x_k, y-y_k)\, m(x,y)\, dxdy,$$

and $\dfrac{\partial \mathcal{F}}{\partial y_k}$ looks like $\dfrac{\partial \mathcal{F}}{\partial x_k}$.

We shall further assume that the function $\varphi(x,y)$ describing the influence or "coverage" of each agent is circularly symmetric, i.e. $\varphi(x,y) = \tilde{\varphi}(r)$ where $\left(r = \sqrt{x^2+y^2}\right)$, hence $\varphi(x,y) = \tilde{\varphi}\left(\sqrt{x^2+y^2}\right)$. Therefore,

$$\frac{\partial \varphi}{\partial x} = \dot{\tilde{\varphi}}\left(\sqrt{x^2+y^2}\right) \frac{x}{\sqrt{x^2+y^2}}, \tag{14}$$

$$\frac{\partial \varphi}{\partial y} = \dot{\tilde{\varphi}}\left(\sqrt{x^2+y^2}\right)\frac{y}{\sqrt{x^2+y^2}}. \tag{15}$$

This result in the following set of equations for the gradient of $\mathcal{F}$ w.r.t. (x_k, y_k) gives us

$$\frac{\partial \mathcal{F}}{\partial x_k} = \tag{16}$$

$$\begin{aligned}
&-2\iint_{\Omega}\Psi(x,y)\frac{(x-x_k)\,\dot{\tilde{\varphi}}\left(\sqrt{(x-x_k)^2+(y-y_k)^2}\right)}{\sqrt{(x-x_k)^2+(y-y_k)^2}}m(x,y)\,dxdy\\
&\quad+2\sum_{i=1}^{N}\iint_{\Omega}\tilde{\varphi}\left(\sqrt{(x-x_i)^2+(y-y_i)^2}\right)\\
&\qquad\times\frac{(x-x_k)\,\dot{\tilde{\varphi}}\left(\sqrt{(x-x_k)^2+(y-y_k)^2}\right)}{\sqrt{(x-x_k)^2+(y-y_k)^2}}m(x,y)\,dxdy.
\end{aligned}$$

Assume that $m(x,y)=1$. Now, using the circular symmetry, we may redefine the x, y coordinates to make x_k, y_k the origin and with every other agent j, define its location to be, for the purpose of integration, $(r_{k-j}, 0)$. This can obviously be done, since $\varphi(x,y)$ is circularly symmetric.

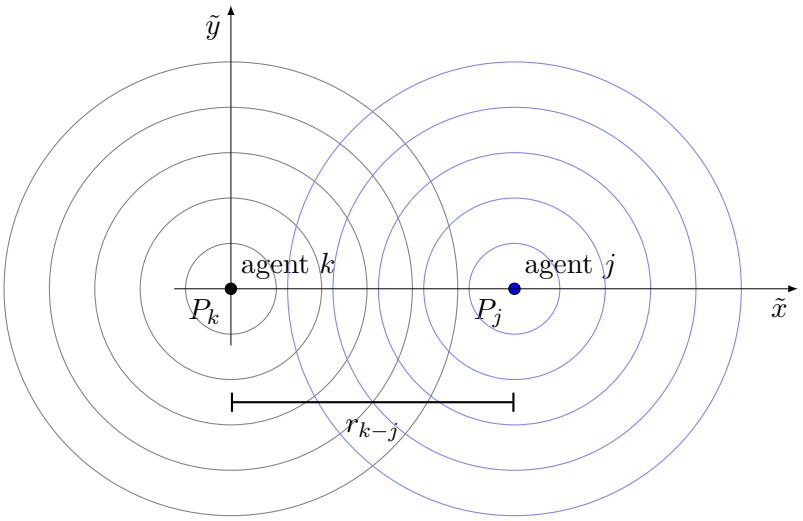


Figure 1: The "view" of agent k, from $(x_k, y_k) = (0,0)$, toward agent j.

With these assumptions (see Figure 1), we have that the integration in $\dfrac{\partial \mathcal{F}}{\partial x_k}$ simplifies to:

$$\frac{\partial \mathcal{F}}{\partial x_k} = \tag{17}$$

$$\begin{aligned}
&-2\iint_{\Omega}\Psi(\tilde{x}+x_k,\tilde{y}+y_k)\,\dot{\tilde{\varphi}}\left(\sqrt{\tilde{x}^2+\tilde{y}^2}\right)\frac{\tilde{x}}{\sqrt{\tilde{x}^2+\tilde{y}^2}}d\tilde{x}d\tilde{y}\\
&+2\textstyle\sum_{j=1}^{N}\iint_{\Omega}\tilde{\varphi}\left(\sqrt{(\tilde{x}+x_k-x_j)^2+(\tilde{y}+y_k-y_j)^2}\right)\dot{\tilde{\varphi}}\left(\sqrt{\tilde{x}^2+\tilde{y}^2}\right)\frac{\tilde{x}}{\sqrt{\tilde{x}^2+\tilde{y}^2}}d\tilde{x}d\tilde{y}\\
&=-2\iint_{\Omega}\Psi(\tilde{x}+x_k,\tilde{y}+y_k)\,\dot{\tilde{\varphi}}\left(\sqrt{\tilde{x}^2+\tilde{y}^2}\right)\frac{\tilde{x}}{\sqrt{\tilde{x}^2+\tilde{y}^2}}d\tilde{x}d\tilde{y}\\
&+2\textstyle\sum_{j=1}^{N}\iint_{\Omega}\tilde{\varphi}\left(\sqrt{(\tilde{x}+r_{k-j})^2+\tilde{y}^2}\right)\dot{\tilde{\varphi}}\left(\sqrt{\tilde{x}^2+\tilde{y}^2}\right)\frac{\tilde{x}}{\sqrt{\tilde{x}^2+\tilde{y}^2}}d\tilde{x}d\tilde{y}
\end{aligned}$$

and $\dfrac{\partial \mathcal{F}}{\partial y_k}$ similarly.

Clearly, we see that

$$\frac{\partial \mathcal{F}}{\partial x_k} = -2A_{x_k}\left\{\Psi\left(\cdot,\cdot\right),\left(x_k,y_k\right)\right\} + 2\sum_{j=1}^{N} R_{x_k}\left\{r_{k-j}\right\} \tag{18}$$

where A_{x_k} is given by

$$A_{x_k} = \iint\limits_{\Omega} \Psi\left(\tilde{x}+x_k,\tilde{y}+y_k\right)\dot{\tilde{\varphi}}\left(\sqrt{\tilde{x}^2+\tilde{y}^2}\right)\frac{\tilde{x}}{\sqrt{\tilde{x}^2+\tilde{y}^2}}d\tilde{x}d\tilde{y} \tag{19}$$

and

$$R_{x_k} = \iint\limits_{\Omega} \tilde{\varphi}\left(\sqrt{\left(\tilde{x}+r_{k-j}\right)^2+\tilde{y}^2}\right)\dot{\tilde{\varphi}}\left(\sqrt{\tilde{x}^2+\tilde{y}^2}\right)\frac{\tilde{x}}{\sqrt{\tilde{x}^2+\tilde{y}^2}}d\tilde{x}d\tilde{y}. \tag{20}$$

The conclusion of this rather long but straightforward argument is that the gradient of the optimization criterion w.r.t. each agent location $P_k \triangleq (x_k, y_k)$ is a vector sum of the form

$$\begin{bmatrix} \dfrac{\partial \mathcal{F}}{\partial x_k} \\ \\ \dfrac{\partial \mathcal{F}}{\partial y_k} \end{bmatrix} = -2\begin{bmatrix} A_{x_k} \\ \\ A_{y_k} \end{bmatrix} + 2\sum_{j=1}^{N}\begin{bmatrix} R_{x_k}\left\{r_{k-j}\right\} \\ \\ R_{y_k}\left\{r_{k-j}\right\} \end{bmatrix} \tag{21}$$

expressing the "interaction" of agent k with the demand $\Psi(x,y)$ at P_k and the interaction of agent k with each of the agents in the swarm (it is easy to see that the "self interaction" terms are zero if the interaction domain Ω is the entire plane $\mathbb{R}^2$).

3 Agent Dynamics

Gradient Descent Optimization for the functional $\mathcal{F}_{\{P_1,P_2,\ldots,P_N\}}$, as discussed in the previous section, yields the dynamical system

$$\frac{dP_k}{dt} = -\gamma\frac{\partial \mathcal{F}}{\partial P_k} = -\gamma\begin{bmatrix} \dfrac{\partial \mathcal{F}}{\partial x_k} \\ \\ \dfrac{\partial \mathcal{F}}{\partial y_k} \end{bmatrix} = 2\gamma\begin{bmatrix} A_{x_k} \\ \\ A_{y_k} \end{bmatrix} - 2\gamma\sum_{j=1}^{N}\begin{bmatrix} R_{x_k}\left\{r_{k-j}\right\} \\ \\ R_{y_k}\left\{r_{k-j}\right\} \end{bmatrix} \tag{22}$$

which involves a demand-driven attraction vector $\mathcal{A}_{\{\Psi,P_k\}}$ and a set of $(N-1)$ nonzero interaction vectors with other agents ("repulsion" terms ensuring that agents will not get "too close" to others).

This gradient descent process requires agents to move at varying speeds determined by the sum of the repulsion and demand-driven attraction vectors.

If we normalize the speed of all agents to make them move with unit speed, we are led to "normalized block-coordinate descent" dynamics, the blocks being the pairs of coordinates (x_k, y_k), i.e. the agent locations.

Recently, using this approach, we considered a problem of decentralized and distributed task allocation. Suppose that the "demand" profile $\Psi(x,y)$ is a superposition of several "task" components located at a set of M known positions $T_l \triangleq (x_{T_l}, y_{T_l}) \mid l = 1,2,\ldots,M$. Assume also that the components of the demand $\Psi(x,y)$ induce on their neighborhood a demand profile of the form $n_l\varphi(x,y)$ where $\varphi(x,y)$ is a bivariate function similar to the "sensing" and coverage profile of the agents in the swarm. Then we have the demand given by

$$\Psi(x,y) = \sum_{l=1}^{M} n_l\varphi\left(x-x_{T_l},y-y_{T_l}\right) + 1_{\Omega}(x,y) \tag{23}$$

in the functional to be optimized where we added the Ω-indicator term $1_{\Omega}(x,y)$ to enforce the agents to operate within the region of interest $\Omega \subset \mathbb{R}^2$.

This leads to a coverage functional to be optimized that looks as follows:

$$\Phi_{\{P_1,P_2,...,P_N\}} = \tag{24}$$

$$\iint\limits_{\Omega} \left(1+\sum_{l=1}^{M} n_l \varphi\left(x-x_{T_l}, y-y_{T_l}\right) - \sum_{j=1}^{N} \varphi\left(x-x_j, y-y_j\right)\right)^2 dxdy.$$

Hence we have a function approximation problem, i.e. we want to closely approximate a sum of M weighted bivariate demand functions with the N bivariate coverage profiles of the moving agents (and we assume $N \gg M$).

As a simple example we may take the $\varphi(x,y)$ profile to be a Gaussian-like "bump" function, i.e. $\varphi(x,y) = e^{-\Lambda\left(x^2+y^2\right)}$ (with $\Lambda > 0$). In this case we are led to a dynamical system moving the "approximating" agents' locations $\{P_i\}_{i=1...N}$ according to the above-mentioned normalized block-coordinate descent equations, explicitly given as follows:

$$\left\{\frac{dP_k}{dt} = u_k(t) = \frac{V_k(t)}{\|V_k(t)\|}\right\}_{k=1...N} \tag{25}$$

with the control vectors $V_k(t)$ given by attraction to tasks terms minus terms representing repulsion by peer agents

$$V_k(t) \triangleq \sum_{l=1}^{M} n_l \bar{A}\left(r_{k-l}\right) - \sum_{j=1}^{N} n_l \bar{R}\left(r_{k-j}\right) \tag{26}$$

with

$$\begin{cases} \bar{A}\left(r_{k-l}\right) = \eta\left(r_{k-l}\right) \dfrac{T_l - P_k}{r_{k-l}} \\ \\ \bar{R}\left(r_{k-j}\right) = \eta\left(r_{k-j}\right) \dfrac{P_j - P_k}{r_{k-j}} \end{cases} \tag{27}$$

where

$$\begin{cases} r_{k-l} \triangleq \|T_l - P_k\| \\ \\ r_{k-j} \triangleq \|P_j - P_k\| \end{cases} \tag{28}$$

and

$$\eta(r) = \frac{1}{2}\pi r e^{-\frac{1}{2}\Lambda r^2}. \tag{29}$$

4 Concluding Remarks

We have seen that least squares function approximation, in a nonstandard context where the set of approximating "basis" functions move in the domain rather than being weighted by scalar gains, is an interesting model for designing multi-agent swarm dynamics. We outlined some interesting initial results in this direction and a wealth of variations are possible.

The fact that least squares optimization leads naturally to various types of attraction and repulsion dynamic systems is quite interesting. Such dynamics are sometimes postulated and analyzed as fundamental in multi-agent robotics research, as well as in many multiparticle potential-based interaction systems in physics and biology.

Many interesting extensions are still under consideration and will be reported in the future. The closest research works in the multi-agent literature to the approach outlined above, as well as our recent results with various extensions considered so far are discussed in two recently posted reports[1, 2].

References

[1] M. Amir, Y. Koifman, Y. Bloch, A. Barel, and A. M. Bruckstein, "Multi-agent distributed and decentralized geometric task allocation," in *2023 62nd IEEE Conference on Decision and Control (CDC)*, pp. 8355–8362, IEEE, Dec. 2023.

[2] Y. Koifman, A. Barel, and A. M. Bruckstein, "Distributed and decentralized control and task allocation for flexible swarms," May 2024. arXiv:2405.13941.



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