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Multiple Granulation Knowledge Acquisition under β-Dominance Relation in Incomplete Ordered Information System

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Abstract

Given that the expanded dominance relation is overly lenient while the limited dominance relation is excessively strict, this paper proposes a new β-dominance relation in incomplete ordered information system, and on this basis, constructs both an optimistic multiple granulation rough set model and a pessimistic multiple granulation rough set model. Subsequently, the paper elaborates in detail on the accuracy measures and approximation qualities of these two multiple granulation rough set models, along with their corresponding algorithms. Finally, through experimental analysis of data from the University of California, Irvine data warehouse, it is confirmed that the proposed multiple granulation rough set models exhibit outstanding performance in both knowledge acquisition and fault tolerance capability.

Keywords: Multiple granulation rough set; β-dominance relation; Incomplete ordered information system; Approximation quality.

1 Introduction

The rough set model was proposed by Pawlak in 1982 as a theory for handling uncertainty issues in datasets [1]. To date, it has been successfully applied across multiple domains including machine learning [2], data mining [3], pattern recognition [4], and group decision-making [5].

As is well known, Pawlak's rough set model is fundamentally based on complete information system and equivalence relation when processing dataset. However, this model proves inadequate when dealing with incomplete information system or non-equivalence relation in complete information system. Moreover, from the perspective of granular computing, a binary relation on the universe U in an information system can be regarded as a granularity, while the corresponding partition can be viewed as a granular structure. Consequently, Pawlak's rough set model represents a single granulation rough set model, exhibiting deficiencies in adaptability, fault tolerance, and flexibility when addressing complex decision-making scenarios. Based on the aforementioned issues, scholars have conducted research on rough sets primarily from two aspects: first, establishing appropriate binary relations in incomplete information system or complete information system with non-equivalence relation, thereby proposing corresponding rough set model. Second, extending single granulation rough set model to multiple granulation rough set model and conducting a series of studies within the theoretical framework of Pawlak's rough set.

When dealing with complete information system with non-equivalence relation or incomplete information system, numerous scholars have conducted in-depth research on the Pawlak's rough set model to expand its application scope, achieving a series of substantial result. In complete information system with non-equivalence relation, Greco employed dominance relation to replace equivalence relation, thereby proposing a rough set model based on dominance relation [6]. Wang defined probabilistic dominance relation in multiple attribute decision-making and accordingly introduced a decision-making model based on probabilistic dominance relation [7]. Further studies can be found in references [8, 9]. In incomplete information system, Kryszkiewicz treated missing value as any value within the domain, and proposed a tolerance relation, thereby introducing a rough set model based on tolerance relation [10]. In system with incomplete information and partial order properties, Shao proposed the expanded dominance relation rough set model[11] , while Luo introduced the limited dominance relation rough set model[12]. Furthermore, Wang defined the prior probability tolerance dominance relation in incomplete fuzzy ordered information system, and based on this relation, proposed a three-way decision-making method grounded in regret theory[13]. Mondal developed a potential dominance relation model within an incomplete ordered information system, leveraging the possibility measure of interval-valued Fermatean connection numbers specific to Broussonetia papyrifera. This innovation facilitated the implementation of three-way multiple attribute decision modeling in such incomplete ordered information system [14]. Based on the TOPSIS method and the concept of regret theory, Li respectively defined the dominance relation in relation to the positive ideal solution and negative ideal solution within the incomplete ordered information system, which allow for the effective classification and ranking of objects [15]. However, in-depth analysis reveals that the expanded dominance relation is overly broad, while the limited dominance relation is excessively stringent. They exhibit significant inadequacies in knowledge acquisition within incomplete ordered information system. Therefore, to enhance the fault tolerance and approximation quality of incomplete ordered information system, defining its dominance relation from a probabilistic perspective undoubtedly constitutes a worthwhile topic.

In multiple granulation rough set model, Qian successfully expanded the single granulation rough set model into the multiple granulation rough set model for the first time, achieving this by incorporating multiple equivalence relations within the universe to precisely define the upper and lower approximation operators. This innovation not only enriched and improved the theoretical framework of multiple granulation rough set, but also significantly enhanced the performance metrics of rough set theory in dealing with complex scenarios, including adaptability, fault tolerance, flexibility, and approximation quality[16]. Therefore, Zhang proposed a multiple granulation intuitionistic fuzzy rough set model in the intuitionistic fuzzy set information system [17]. Zhan introduced the covering multiple granulation fuzzy rough set model in the covering information system and discussed its properties and applications [18]. Xu proposed a local generalized neighborhood multiple granulation rough set model, and then analyzed the dynamic updating process of global multiple granulation rough set and local multiple granulation rough set when objects change [19]. Meanwhile, Li introduced a local multiple granulation rough set model in the complete ordered information system, and further provided dynamic object approximation updating algorithms for both global and local multiple granulation rough sets [20]. Furthermore, Yang extended the three binary relations of tolerance relation, limited tolerance relation, and similarity relation in incomplete information system to the multiple granulation rough set models and investigated their properties [21]. Additionally, Zhai proposed multiple granulation rough set models based on expanded dominance relation and limited dominance relation in incomplete ordered information system [22]. In the complete information system, by introducing the rough membership function and approximation parameter from variable precision rough set into the multiple granulation environment, Chen proposed a variable precision multiple granulation rough set model with misclassification tolerance and noise resistance [23]. More detailed studies on multiple granulation rough sets can be found in references [24, 25, 26, 27]. However, research has found that the multiple granulation rough set model based on expanded dominance relation is overly lenient in terms of knowledge acquisition and fault tolerance, while the multiple granulation rough set model with limited dominance relation is overly strict in these aspects, in incomplete ordered information system. Both models lack misclassification tolerance and noise resistance in data processing. Therefore, it is necessary to improve the multiple granulation rough set model in incomplete ordered information system, thereby enhancing their fault tolerance capability and expanding their application scope.

The main contributions of this paper are as follows: (1) From a probabilistic perspective, we define the dominance relation between two objects under single attribute in an incomplete ordered information system, and propose a discrimination method for two objects under an attribute set, namely the β-dominance relation, laying the foundation for constructing the multiple granulation rough set model. (2) By incorporating the β-dominance relation into a multiple granulation context, we develop both an optimistic multiple granulation rough set model and a pessimistic multiple granulation rough set model. These models are designed with capabilities for misclassification tolerance and noise resistance, all grounded in the β-dominance relation. Moreover, we conduct an in-depth study on the properties of the proposed models, and explore the relationships among three multiple granulation rough set models under the β-dominance relation, limited dominance relation, and expanded dominance relation. (3) We design algorithms for the β-dominance class, accuracy measure, and approximation quality in the optimistic multiple granulation rough set under the β-dominance relation. The design of these three algorithms in the pessimistic multiple granulation rough set model is similar.

The remaining sections of this paper are organized as follows: Section 2 reviews expanded dominance relation, limited dominance relation, and the fundamental concepts of multiple granulation rough set models based on these two dominance relations. Section 3 delves into the concept and characteristics of β-dominance relation, as well as the relevant algorithm for calculating object dominance classes. Section 4 delves into the concepts and properties of both optimistic multiple granulation rough set model and pessimistic multiple granulation rough set model under the β-dominance relation. Furthermore, the accuracy measures and approximation qualities of these two models are proposed, along with the corresponding algorithms. Section 5 explores the relationships among multiple granulation rough set models based on β-dominance relation, expanded dominance relation, and limited dominance relation. Section 6 demonstrates the superiority of the proposed models through experimental analysis. Section 7 summarizes the research findings of this paper and points out future research directions.

2 Preliminaries

In this section, we briefly review the concepts of expanded dominance relation and limited dominance relation, and explore some definitions of multiple granulation rough set models based on two dominance relations in incomplete ordered information system.

2.1 Dominance relation in incomplete ordered information system

Let an information system be described as $IS = (U, AT, V, f)$, and it is a quadruple, both U and AT are finite sets that are not empty, which contain n objects and m attributes, respectively. AT is called condition attributes. V_a is a value of the object in single attribute a, $V = \bigcup_{a\in AT} V_a$. f is a mapping from object to attribute, represented by $f : U \times a \longrightarrow V_a$, $f(\psi, a) \in V_a$, $\forall a \in AT$, $\forall \psi \in U$, called an information function. If the value of the object in information system is missing, represented by $f(\psi, a) = *$, then it is called an incomplete information system, if not, it is a complete information system [1, 10]. If the attributes exhibit partial order relationship under the incomplete information system, then this incomplete information system is referred to as incomplete ordered information system, denoted as $IOIS$ [11, 12].

Definition 2.1 [11] Let $IOIS = (U, AT, V, f)$, $A \subseteq AT$, $\forall \psi_1, \psi_2 \in U$, we have that

$$R_A^{\geq *E} = \{(\psi_1, \psi_2) \in U \times U | \forall a \in A, f(\psi_1, a) \geq f(\psi_2, a) \vee f(\psi_1, a) = * \vee f(\psi_2, a) = *\},$$

where the logical operator "or" is described by " $\vee$ ". Denote the set $R_A^{\geq *E}$ as a dominance relation under $IOIS$, called expanded dominance relation, denoted as EDR. Represented by $[\psi_1]_A^{\geq *E} = \{\psi_2 \in U | (\psi_2, \psi_1) \in R_A^{\geq *E}\} = \{\psi_2 \in U | f(\psi_2, a) \geq f(\psi_1, a) \vee f(\psi_1, a) = * \vee f(\psi_2, a) = *, \forall a \in A\}$, then expanded dominance class about ψ_1 is expressed as $[\psi_1]_A^{\geq *E}$. Because of $U/R_A^{\geq *E} = \{\psi_1]_A^{\geq *E} | \forall \psi_1 \in U\}$, therefore $U/R_A^{\geq *E}$ forms a cover in U about attribute set A.

From definition 2.1, the EDR indicates that the missing value in the $IOIS$ can be regarded as any value, and is prone to amplifying dominance class of objects. For example, $\psi_1 = (1, 1, *)$, $\psi_2 = (1, 1, 5)$, $\psi_3 = (*, *, 5)$, where suppose the maximum value and the minimum value under each attribute are 5 and 1, respectively. It can get the expanded dominance class of each object as $[\psi_1]_A^{\geq *E} = \{\psi_1, \psi_2, \psi_3\}$, $[\psi_2]_A^{\geq *E} = \{\psi_1, \psi_2, \psi_3\}$, $[\psi_3]_A^{\geq *E} = \{\psi_1, \psi_2, \psi_3\}$. However, from probability perspective, the probabilities that ψ_1 and ψ_2 are superior to ψ_3 are extremely low. Therefore, a new dominance relation is brought forward to deal with the defect of the EDR in the literature [12].

Definition 2.2 [12] Let $IOIS = (U, AT, V, f)$, $A \subseteq AT$, $\forall \psi_1, \psi_2 \in U$, we have that

$$R_A^{\geq *L} = \{(\psi_1, \psi_2) \in U \times U | \forall a \in A, f(\psi_1, a) \geq f(\psi_2, a) \vee (f(\psi_1, a) = \max V_a \wedge f(\psi_2, a) = *)$$

$$\vee (f(\psi_1, a) = * \wedge f(\psi_2, a) = \min V_a)\} \cup I_U,$$

where the logical operator "and" is described by " $\wedge$ ", $I_U = \{(\psi_1, \psi_1) | \psi_1 \in U\}$ is reflexive. Denote the set $R_A^{\geq *L}$ as a dominance relation under $IOIS$, called limited dominance relation, denoted as LDR. Represented by $[\psi_1]_A^{\geq *L} = \{\psi_2 \in U | (\psi_2, \psi_1) \in R_A^{\geq *L}\}$, then limited dominance class about ψ_1 is expressed as $[\psi_1]_A^{\geq *L}$. Because of $U/R_A^{\geq *L} = \{[\psi_1]_A^{\geq *L} | \forall \psi_1 \in U\}$, therefore $U/R_A^{\geq *L}$ forms a cover in U about attribute set A.

According to the above definition 2.2, the partition of the LDR is overly strict. For example, $\psi_1 = (5, 4, 2, 1)$, $\psi_2 = (*, *, 2, 1)$, where suppose the maximum value and the minimum value under each attribute are 5 and 1, respectively. It can get the limited dominance class of object ψ_2 as $[\psi_2]_A^{\geq *L} = \{\psi_2\}$. However, from probability perspective, the probability that ψ_1 is superior to ψ_2 is very large. Based on the above motivation, in order to improve the accuracy measure and approximation quality, we must construct a relatively reasonable binary dominance relationship in the $IOIS$.

2.2 Multiple granulation rough set models in $IOIS$

Definition 2.3 [22] Let $IOIS = (U, AT, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, for any $\forall \Psi \subseteq U$. Based on EDR and LDR, the upper and lower approximation operators of optimistic multiple granulation rough sets and pessimistic multiple granulation rough sets for the object set Ψ under the attribute sets $A_1, A_2, ..., A_s$ are respectively defined as follows:

$$\underline{\sum_{i=1}^{s} A_{iE}^{O}}(\Psi) = \{\psi \in U | \vee_{i=1}^{s} [\psi]_{A_i}^{\geq *E} \subseteq \Psi\}, \ \overline{\sum_{i=1}^{s} A_{iE}^{O}}(\Psi) = \{\psi \in U | \wedge_{i=1}^{s} [\psi]_{A_i}^{\geq *E} \cap \Psi \neq \emptyset\};$$

$$\underline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi) = \{\psi \in U | \vee_{i=1}^{s} [\psi]_{A_i}^{\geq *L} \subseteq \Psi\}, \ \overline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi) = \{\psi \in U | \wedge_{i=1}^{s} [\psi]_{A_i}^{\geq *L} \cap \Psi \neq \emptyset\}.$$

$$\underline{\sum_{i=1}^{s} A_{iE}^{P}}(\Psi) = \{\psi \in U | \wedge_{i=1}^{s} [\psi]_{A_i}^{\geq *E} \subseteq \Psi\}, \ \overline{\sum_{i=1}^{s} A_{iE}^{P}}(\Psi) = \{\psi \in U | \vee_{i=1}^{s} [\psi]_{A_i}^{\geq *E} \cap \Psi \neq \emptyset\},$$

$$\underline{\sum_{i=1}^{s}} A_{iL}^{P}(\Psi) = \{\psi \in U | \wedge_{i=1}^{s} [\psi]_{A_i}^{\geq * L} \subseteq \Psi\},\ \overline{\sum_{i=1}^{s}} A_{iL}^{P}(\Psi) = \{\psi \in U | \vee_{i=1}^{s} [\psi]_{A_i}^{\geq * L} \cap \Psi \neq \emptyset\}.$$

$[\underline{\sum_{i=1}^{s} A_{iE}^{O}}(\Psi), \overline{\sum_{i=1}^{s} A_{iE}^{O}}(\Psi)]$ and $[\underline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi), \overline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi)]$ are respectively called optimistic multiple granulation rough set models under EDR and LDR in multiple granulation spaces $A_1, A_2, \cdots, A_s$. optimistic multiple granulation rough set model is denoted as $OMGRSM$.

$[\underline{\sum_{i=1}^{s} A_{iE}^{P}}(\Psi), \overline{\sum_{i=1}^{s} A_{iE}^{P}}(\Psi)]$ and $[\underline{\sum_{i=1}^{s} A_{iL}^{P}}(\Psi), \overline{\sum_{i=1}^{s} A_{iL}^{P}}(\Psi)]$ are respectively called pessimistic multiple granulation rough set models under EDR and LDR in multiple granulation spaces $A_1, A_2, \cdots, A_s$. pessimistic multiple granulation rough set model is denoted as $PMGRSM$.

The accuracy measures of Ψ in $OMGRSM$ and $PMGRSM$ on the basis of EDR and LDR are defined by using the following method. Where is $|\Psi| \neq \emptyset$.

$$\alpha_E^O(\sum_{i=1}^{s} A_i, \Psi) = \frac{|\underline{\sum_{i=1}^{s} A_{iE}^{O}}(\Psi)|}{|\overline{\sum_{i=1}^{s} A_{iE}^{O}}(\Psi)|}, \alpha_L^O(\sum_{i=1}^{s} A_i, \Psi) = \frac{|\underline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi)|}{|\overline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi)|},$$
$$\alpha_E^P(\sum_{i=1}^{s} A_i, \Psi) = \frac{|\underline{\sum_{i=1}^{s} A_{iE}^{P}}(\Psi)|}{|\overline{\sum_{i=1}^{s} A_{iE}^{P}}(\Psi)|}, \alpha_L^P(\sum_{i=1}^{s} A_i, \Psi) = \frac{|\underline{\sum_{i=1}^{s} A_{iL}^{P}}(\Psi)|}{|\overline{\sum_{i=1}^{s} A_{iL}^{P}}(\Psi)|}.$$

Furthermore, let an incomplete ordered decision system be described as $IODS = (U, AT \cup d, V, f)$, and it is also a quadruple, AT is called condition attribute, d is called decision attribute, and $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\{D_1, D_2, \cdots, D_k\}$ is partition of decision class, which is described by d. Based on EDR and LDR, the approximation qualities of the $OMGRSM$ and $PMGRSM$ with respect to the decision attribute d under the attribute sets $A_1, A_2, ..., A_s$ are respectively defined in the following way:

$$\gamma_E^O(\sum_{i=1}^{s} A_i, d) = \frac{\sum_{l=1}^{k}(|\underline{\sum_{i=1}^{s} A_{iE}^{O}}(D_l)|)}{|U|}, \gamma_L^O(\sum_{i=1}^{s} A_i, d) = \frac{\sum_{l=1}^{k}(|\underline{\sum_{i=1}^{s} A_{iL}^{O}}(D_l)|)}{|U|},$$
$$\gamma_E^P(\sum_{i=1}^{s} A_i, d) = \frac{\sum_{l=1}^{k}(|\underline{\sum_{i=1}^{s} A_{iE}^{P}}(D_l)|)}{|U|}, \gamma_L^P(\sum_{i=1}^{s} A_i, d) = \frac{\sum_{l=1}^{k}(|\underline{\sum_{i=1}^{s} A_{iL}^{P}}(D_l)|)}{|U|}.$$

Where $R_d = \{(\psi_1, \psi_2) \in U \times U | g(\psi_1, d) = g(\psi_2, d)\}$, $U/d = \{[\psi_1]_d | \forall \psi_1 \in U\} = \{D_1, D_2, \cdots, D_k\}$, $\gamma_E^O(\sum_{i=1}^{s} A_i, d)$, $\gamma_L^O(\sum_{i=1}^{s} A_i, d)$, $\gamma_E^P(\sum_{i=1}^{s} A_i, d)$ and $\gamma_L^P(\sum_{i=1}^{s} A_i, d)$ are also respectively called the degree of dependency in $OMGRSM$ and $PMGRSM$ on the basis of EDR and LDR, and they are used to evaluate the accuracy of decision classes.

3 β-dominance relation

In this section, according to analyze the deficiencies of EDR and LDR, we put forward a new β-dominance relation, analyze its properties, design an algorithm to compute the β-dominance class for each object, and discuss the differences and relationships among β-dominance relation, EDR, and LDR.

In the $IOIS$, attribute domain of the object can be distributed in many ways, such as uniform distribution, triangular distribution, exponential distribution, and so on. For the convenience of the following discussion, we assume that the attribute values of the $IOIS$ are uniform distribution.

Definition 3.1 Let $IOIS = (U, AT, V, f)$, $\forall a \in AT$, $V_a = \{a_1, a_2, \cdots, a_n\}$ represents the value of the object under single attribute a, respectively, and $a_1 < a_2 < \cdots < a_n$. For any $\forall \psi_1, \psi_2 \in U$, then the probability that ψ_1 is superior to ψ_2 under single attribute a is defined:

$$R_a(\psi_1, \psi_2) = \begin{cases} 1, & f(\psi_1, a) \geq f(\psi_2, a) \\ V_a'(\psi_1)/n, & f(\psi_1, a) \neq * \wedge f(\psi_2, a) = * \\ (n - V_a'(\psi_2) + 1)/n, & f(\psi_1, a) = * \wedge f(\psi_2, a) \neq * \\ (n+1)/2n, & f(\psi_1, a) = * \wedge f(\psi_2, a) = * \\ 0, & f(\psi_1, a) < f(\psi_2, a) \end{cases}$$

where $V_a^{'}(\psi_i)$ means the conversion value of ψ_i under single attribute a, and $\psi_i \in U$. Let $a_1 = 1, a_2 = 2, \cdots, a_n = n$, $V_a^{'}(\psi_i) = \{1, 2, \cdots, n\}$.

The first case and the fifth case in the above definition 3.1 formula are obvious. For the second case, the third case and the fourth case, it can be explained as follows:

(b) the second case: when $f(\psi_1, a) \neq * \wedge f(\psi_2, a) = *$, suppose $f(\psi_1, a) = a_i$, and $V_a^{'}(\psi_1) = i$ indicates that the value of the object ψ_1 under the attribute a is sorted in the i position ($i = 1, 2, ..n$). In this case, the value of the object ψ_2 under the attribute a is unknown or uncertain, that is, its value may be any one of n, then its value sorting can be expressed as $P\{V_a^{'}(\psi_2) = j\} = 1/n$ ($j = 1, 2, ..n$). We can derive:

$$R_a(\psi_1, \psi_2) = P\{f(\psi_1, a) \geq f(\psi_2, a)\}$$
$$= P\{V_a^{'}(\psi_1) \geq V_a^{'}(\psi_2)\} = P\{V_a^{'}(\psi_2) \leq i\}$$
$$= \textstyle\sum_{j=1}^{i} P\{V_a^{'}(\psi_2) = j\} = i/n = V_a^{'}(\psi_1)/n.$$

(c) the third case: when $f(\psi_1, a) = * \wedge f(\psi_2, a) \neq *$, suppose $f(\psi_2, a) = a_j$, and $V_a^{'}(\psi_2) = j$ indicates that the value of the object ψ_2 under the attribute a is sorted in the j position ($j = 1, 2, ..n$). In this case, the value of the object ψ_1 under the attribute a is unknown or uncertain, that is, its value may be any one of n, then its value sorting can be expressed as $P\{V_a^{'}(\psi_1) = i\} = 1/n$ ($i = 1, 2, ..n$). We can derive:

$$R_a(\psi_1, \psi_2) = P\{f(\psi_1, a) \geq f(\psi_2, a)\}$$
$$= P\{V_a^{'}(\psi_1) \geq V_a^{'}(\psi_2)\} = P\{V_a^{'}(\psi_2) \geq j\}$$
$$= \textstyle\sum_{i=j}^{n} P\{V_a^{'}(\psi_1) = i\} = (n - j + 1)/n = (n - V_a^{'}(\psi_2) + 1)/n.$$

(d) the fourth case: when $f(\psi_1, a) = * \wedge f(\psi_2, a) = *$, and the values of the objects ψ_1 and ψ_2 under the attribute a are unknown or uncertain, that is, their values may be any of n, then their values sorting can be expressed as $P\{V_a^{'}(\psi_1) = i\} = 1/n$ and $P\{V_a^{'}(\psi_2) = j\} = 1/n$ ($i = 1, 2, ..n; j = 1, 2, ..n$). We can derive:

$$R_a(\psi_1, \psi_2) = P\{f(\psi_1, a) \geq f(\psi_2, a)\} = P\{V_a^{'}(\psi_1) \geq V_a^{'}(\psi_2)\}$$
$$= P\{i \geq j | V_a^{'}(\psi_1) = i, V_a^{'}(\psi_1) = j\} = \textstyle\sum_{i=1}^{n}(P\{V_a^{'}(\psi_1) = i\} \sum_{j=1}^{i}(P\{V_a^{'}(\psi_2) = j\}))$$
$$= \textstyle\sum_{i=1}^{n}(1/n \sum_{j=1}^{i} 1/n) = \sum_{i=1}^{n}(i/n^2) = n(n+1)/2n^2 = (n+1)/2n.$$

Definition 3.1 tells us the dominance probability between two objects under single attribute in the $IOIS$. The dominance probability between two objects under the attribute set is as follows.

Definition 3.2 Let $IOIS = (U, AT, V, f)$, and $A \subseteq AT$, $\forall \psi_1, \psi_2 \in U$, we have that

$$R_A^{\geq *\beta} = \{(\psi_1, \psi_2) \in U \times U | R_a(\psi_1, \psi_2) \geq \beta\ , \forall a \in A\} \cup I_U.$$

The set $R_A^{\geq *\beta}$ is called β-dominance relation under $IOIS$, and $0 < \beta \leq 1$. β-dominance class of ψ_1 is denoted by using $[\psi_1]_A^{\geq *\beta}$, and $[\psi_1]_A^{\geq *\beta} = \{\psi_2 \in U | (\psi_2, \psi_1) \in R_A^{\geq *\beta}\}$. Because of $U/R_A^{\geq *\beta} = \{[\psi_1]_A^{\geq *\beta} | \forall \psi_1 \in U\}$, therefore $U/R_A^{\geq *\beta}$ forms a cover in U about attribute set A.

To facilitate obtaining the β-dominance class for each object, the detailed steps are provided in Algorithm 1,and its time complexity is given as follows. Steps 4 to 8 assign the value $*$ to objects with missing values under single attribute, with the time complexity of $O(|U|)$; Steps 9 to 11 discretize the values of the incomplete ordered information system under single attribute, also with the time complexity of $O(|U|)$. Steps 12 to 34 calculate the β-dominance class of each object under single attribute by using definition 3.1 and 3.2, and its time complexity is $O(|U|^2)$. Step 35 calculates the β-dominance class of each object under the attribute set by using definition 3.2, and its time complexity is $O(|AT||U|^2)$. Therefore, the time complexity for all object dominance classes is $O(|AT||U|^2)$ in the Algorithm 1.

Proposition 3.1 Let $IOIS = (U, AT, V, f)$, $A, B \subseteq AT$, for any $\forall \psi \in U$, the following properties are established:

(1) $R_A^{\geq *\beta}$ satisfies reflective, but it dissatisfies symmetric and transitive,

Algorithm 1: An algorithm for computing β-dominance classes of each object in $IOIS$

```
Input  : An IOIS = (U, AT, V, f) , A_i ⊆ AT, ∀a ∈ A_i, threshold β ∈ (0,1] ;
Output : β-dominance classes of each object, ∀ψ_j ∈ U, [ψ_j]^{≥*β}_{A_i}.
begin
    Initialize [ψ_j]^{≥*β}_{A_i} ← ∅, set β ∈ (0,1].
    for each a ∈ A_i do
        for j = 1 to n do
            if f(ψ_j, a) = φ then
                f(ψ_j, a) = *;
        for j = 1 to n do
            Computef(ψ_j, a) hash value f'(ψ_j, a), f'(ψ_j, a) ∈ {1, 2, ..., m}(m ≤ n),
             minf(ψ_j, a) = 1, maxf(ψ_j, a) = m;
        for h = 1 to n do
            for j = 1 to n do
                if h = j then
                    continue;
                if f(ψ_h, a) ≠ φ,f(ψ_j, a) ≠ φ ,and f(ψ_h, a) ≥ f(ψ_j, a) then
                    R_a(ψ_h, ψ_j) = 1;
                if f(ψ_h, a) ≠ φ,f(ψ_j, a) ≠ φ ,and f(ψ_h, a) < f(ψ_j, a) then
                    R_a(ψ_h, ψ_j) = 0;
                if f(ψ_h, a) ≠ φ,f(ψ_j, a) = φ then
                    R_a(ψ_h, ψ_j) = f'(ψ_h, a)/m;
                if f(ψ_h, a) = φ,f(ψ_j, a) ≠ φ then
                    R_a(ψ_h, ψ_j) = (m − f'(ψ_j, a) + 1)/m;
                if f(ψ_h, a) = φ,f(ψ_j, a) = φ then
                    R_a(ψ_h, ψ_j) = (m + 1)/2m;
    ∀ψ_j ∈ U, [ψ_j]^{≥*β}_{A_i} = {ψ_h | R_a(ψ_h, ψ_j) ≥ β, ∀a ∈ A_i} ⋃ {ψ_j}.
return: ∀ψ_j ∈ U, [ψ_j]^{≥*β}_{A_i}.
```

(2) For $B \subseteq A \subseteq AT$, then $R_{AT}^{\geq*\beta} \subseteq R_{A}^{\geq*\beta} \subseteq R_{B}^{\geq*\beta}$ and $[\psi]_{AT}^{\geq*\beta} \subseteq [\psi]_{A}^{\geq*\beta} \subseteq [\psi]_{B}^{\geq*\beta}$,

(3) For $0 < \beta_1 \leq \beta_2 \leq 1$, then $[\psi]_{A}^{\geq*\beta_2} \subseteq [\psi]_{A}^{\geq*\beta_1}$,

(4) For $0 < \beta \leq 1$, then $[\psi]_{A}^{\geq*L} \subseteq [\psi]_{A}^{\geq*\beta} \subseteq [\psi]_{A}^{\geq*E}$.

Proof. Combine the above definitions, they are easily proved.

According to Proposition 3.1, we draw conclusions in three aspects. First, the larger the attribute set, the smaller the β-dominance class of the object; second, for the same object, when β is within the range of [0,1], the larger its value, the smaller the corresponding dominance class; third, for the same object, the size of the β-dominance class set lies between the limited dominance class and the expanded dominance class.

Example 1 Suppose that the values of three objects under single attribute a are $f(\psi_1, a) = 3$, $f(\psi_2, a) = *$, $f(\psi_3, a) = 4$ in the $IOIS$, respectively. And their smallest attribute value is 1, and their largest attribute value is 6.

Suppose $\beta = 0.5$ and $|a| = 1$, from definition 3.1, we can get that $R_a(\psi_1, \psi_2) = 3/6$, $R_a(\psi_2, \psi_3) = 3/6$. But $R_a(\psi_1, \psi_3) = 0$. It can be demonstrated that $R_{a}^{\geq*\beta}$ does not satisfy transitive.

In order to research the relationships among β-dominance relation, DER and LDR, $\forall a \in A$, suppose n indicates the maximum value of the conversion value $V_a^{'}(\psi)$, and the following conclusions can be drawn from Figure 1.

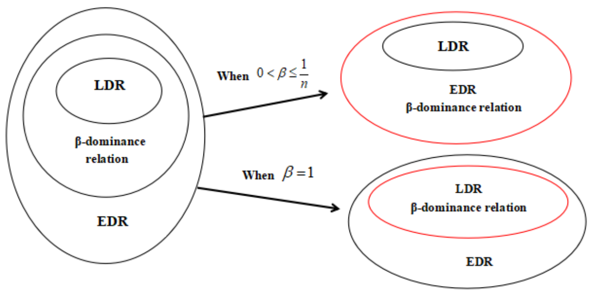

Figure 1: The relationships among DER ,LDR and β-dominance relation

(1) For $0 < \beta \leq 1/n$, then $[\psi]_{A}^{\geq*\beta} = [\psi]_{A}^{\geq*E}$, i.e, β-dominance relation is transformed into EDR;

(2) For $\beta = 1$, then $[\psi]_{A}^{\geq*\beta} = [\psi]_{A}^{\geq*L}$, i.e, β-dominance relation is transformed into LDR.

It is shown that β-dominance relation is an extension of EDR and LDR in the $IOIS$. In other words, β-dominance relation is more adaptable and flexible to noise data.

Similar to the Pawlak's rough set model[1], we propose relevant concepts of single granulation rough set model based on the β-dominance relation.

Definition 3.3 Let $IOIS = (U, AT, V, f)$, $A \subseteq AT$, $\Psi \subseteq U$, for any $\forall\psi \subseteq U$, the two approximation operators about upper and lower of Ψ on A under β-dominance relation are defined:

$$\underline{R_{A}^{\beta}}(\Psi) = \{\psi \in U | [\psi]_{A}^{\geq*\beta} \subseteq \Psi\}, \overline{R_{A}^{\beta}}(\Psi) = \{\psi \in U | [\psi]_{A}^{\geq*\beta} \cap \Psi \neq \emptyset\}.$$

$[\underline{R_{A}^{\beta}}(\Psi), \overline{R_{A}^{\beta}}(\Psi)]$ is called single granulation rough set model under β-dominance relation. Moreover, if $\underline{R_{A}^{\beta}}(\Psi) = \overline{R_{A}^{\beta}}(\Psi)$, and Ψ is a definable set about attribute set A. Otherwise, Ψ is a rough set.

The uncertainty or boundary area of Ψ under single granulation rough set model of β-dominance relation is defined by using the following method.

$$BND_{A}^{\beta}(\Psi) = \overline{R_{A}^{\beta}}(\Psi) - \underline{R_{A}^{\beta}}(\Psi).$$

4 Multiple granulation rough set models under β-dominance relation

We can understand that a β-dominance relation can be regarded as single granulation, while multiple β-dominance relations can be viewed as multiple granulation. In expert evaluation system, the following two scenarios may occur: first, the event is deemed to pass as long as any one condition in the system is met; second, all conditions in the system must be met for the event to pass. Therefore, from the perspective of granular computing, they have the following four conditions:

(a)At least one granulation is included in a given concept;

(b)All the granulation intersecting a given concept is not equal to an empty set;

(c)All the granulation is contained in the given concept;

(d)At least one granulation intersecting a given concept is not equal to an empty set.

We can use (a) and (b) to describe the lower and upper approximations under β-dominance relation in optimistic situation, and (c) and (d) as the lower and upper approximations under β-dominance relation in pessimistic situation.

4.1 Optimistic multiple granulation rough set model

In this subsection, we propose for the first time an optimistic multiple granulation rough set model based on the β-dominance relation in the $IOIS$, and its related properties are studied.

Definition 4.1 Let $IOIS = (U, AT, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, for any $\forall \Psi \subseteq U$. The lower and upper approximation operators of Ψ for optimistic multiple granulation of β-dominance relation under attribute sets $A_1, A_2, ..., A_s$ are respectively defined as:

$$\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) = \{\psi \in U | \vee_{i=1}^{s} [\psi]_{A_i}^{\geq *\beta} \subseteq \Psi\}, \quad \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) = \{\psi \in U | \wedge_{i=1}^{s} [\psi]_{A_i}^{\geq *\beta} \cap \Psi \neq \phi\}.$$

$[\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi), \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)]$ is called $OMGRSM$ under β-dominance relation in multiple granulation spaces $A_1, A_2, \cdots, A_s$. Moreover, if $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) = \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$, then Ψ is a definable set about multiple granulation space $A_1, A_2, \cdots, A_s$. If not, Ψ is a rough set in the $IOIS$.

The uncertainty or boundary area of Ψ under $OMGRSM$ of β-dominance relation is defined by using the following method.

$$\sum_{i=1}^{s} BND_{i\beta}^{O}(\Psi) = \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) - \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi).$$

It can be inferred from definition 4.1 that the two approximation operators for lower and upper approximations in $OMGRSM$ are obtained by utilizing the object dominance classes derived from multiple β-dominance relations. If all granulation spaces are equal ($A_1 = A_2 = \cdots = A_s$), then $OMGRSM$ under β-dominance relation will become single granulation rough set model under β-dominance relation.

According to definition 4.1, the following properties of $OMGRSM$ under β-dominance relation are got by us in the $IOIS$.

Proposition 4.1 Let $IOIS = (U, AT, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\forall \Psi \subseteq U$, the following properties in $OMGRSM$ under β-dominance relation are established:

(1) $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \subseteq \Psi \subseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$,

(2) $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\emptyset) = \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\emptyset) = \emptyset$, $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(U) = \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(U) = U$,

(3) $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\sim \Psi) = \sim \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi), \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\sim \Psi) = \sim \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$,

(4) $\beta_1 \leq \beta_2 \Rightarrow \underline{\sum_{i=1}^{s} A_{i\beta_1}^{O}}(\Psi) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta_2}^{O}}(\Psi)$,

(5) $\beta_1 \leq \beta_2 \Rightarrow \overline{\sum_{i=1}^{s} A_{i\beta_2}^{O}}(\Psi) \subseteq \overline{\sum_{i=1}^{s} A_{i\beta_1}^{O}}(\Psi)$.

Proof. (1a) For any $\forall \psi \in \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$, by definition 4.1, exists $\exists i \in \{1, 2, \cdots, s\}$, we obtain that $[\psi]_{A_i}^{\geq *\beta} \subseteq \Psi$. For any $\forall i \in \{1, 2, \cdots, s\}$, Due to $\psi \in [\psi]_{A_i}^{\geq *\beta}$, so $\psi \in \Psi$. Then $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \subseteq \Psi$.

(1b) $\forall \psi \in \Psi$, for any $\forall i \in \{1, 2, \cdots, s\}$, and $\psi \in [\psi]_{A_i}^{\geq *\beta}$, by definition 4.1, we can have $[\psi]_{A_i}^{\geq *\beta} \cap \Psi \neq \emptyset$. So $\psi \in \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$. Then $\Psi \subseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$.

(2a) From(1), we can obtain that $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\emptyset) \subseteq \emptyset$, but $\emptyset \subseteq \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\emptyset)$. So $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\emptyset) = \emptyset$.

(2b) Assume $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\emptyset) \neq \emptyset$, and exists $\exists \psi$ makes $\psi \in \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\emptyset)$, by definition 4.1, For any $\forall i \in \{1, 2, \cdots, s\}$, we can obtain $[\psi]_{A_i}^{\geq * \beta} \cap \emptyset \neq \emptyset$. But $[\psi]_{A_i}^{\geq * \beta} \cap \emptyset \neq \emptyset$, contradict the hypothesis. So $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\emptyset) = \emptyset$.

(2c) From(1), we can obtain that $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(U) \subseteq U$. Due to $\psi \in U$, For any $\forall i \in \{1, 2, \cdots, s\}$, it is described that $[\psi]_{A_i}^{\geq * \beta} \subseteq U$, so $\psi \in \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(U)$, i.e, $U \subseteq \underline{R_{(A+B)\beta}^{O}}(U)$. Then $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(U) = U$.

(2d) From(1), we can obtain that $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(U) \supseteq U$. But $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(U) \subseteq U$, so $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(U) = U$.

(3a) For any $\forall \psi \in \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\sim \Psi) \Leftrightarrow [\psi]_{A_1}^{\geq * \beta} \subseteq \sim \Psi \vee [\psi]_{A_2}^{\geq * \beta} \subseteq \sim \Psi \vee \cdots \vee [\psi]_{A_s}^{\geq * \beta} \subseteq \sim \Psi \Leftrightarrow [\psi]_{A_1}^{\geq * \beta} \cap \Psi = \emptyset \wedge [\psi]_{A_2}^{\geq * \beta} \cap \Psi = \emptyset \wedge \cdots [\psi]_{A_s}^{\geq * \beta} \cap \Psi = \emptyset \Leftrightarrow \psi \notin \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \Leftrightarrow \psi \in \sim \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$.

(3b) For any $\forall \psi \in \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \Leftrightarrow [\psi]_{A_1}^{\geq * \beta} \subseteq \Psi \vee [\psi]_{A_2}^{\geq * \beta} \subseteq \Psi \vee \cdots \vee [\psi]_{A_s}^{\geq * \beta} \subseteq \Psi \Leftrightarrow [\psi]_{A_1}^{\geq * \beta} \cap \sim \Psi = \emptyset \wedge [\psi]_{A_2}^{\geq * \beta} \cap \sim \Psi = \emptyset \wedge \cdots \wedge [\psi]_{A_s}^{\geq * \beta} \cap \sim \Psi = \emptyset \Leftrightarrow \psi \notin \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\sim \Psi) \Leftrightarrow \psi \in \sim \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\sim \Psi)$.

(4) For any $\forall \psi \in U$,$\forall i \in \{1, 2, \cdots, s\}$, $A_i \subseteq AT$, $\beta_1 \leq \beta_2$, by proposition 3.1, it is described that $[\psi]_{A_i}^{\geq * \beta_2} \subseteq [\psi]_{A_i}^{\geq * \beta_1}$. Combine with definition 4.1 about the concept of the lower approximation operator, we can have $\underline{\sum_{i=1}^{s} A_{i\beta_1}^{O}}(\Psi) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta_2}^{O}}(\Psi)$.

(5) For any $\forall \psi \in U$,$\forall i \in \{1, 2, \cdots, s\}$, $A_i \subseteq AT$, $\beta_1 \leq \beta_2$, by proposition 3.1, it is described that $[\psi]_{A_i}^{\geq * \beta_2} \subseteq [\psi]_{A_i}^{\geq * \beta_1}$. Combine with definition 4.1 about the concept of the upper approximation operator, we can have $\overline{\sum_{i=1}^{s} A_{i\beta_2}^{O}}(\Psi) \subseteq \overline{\sum_{i=1}^{s} A_{i\beta_1}^{O}}(\Psi)$.

The proposition was proved.

Proposition 4.1 demonstrates the correlation between the upper and lower approximation operators and the given object set in the $OMGRSM$ based on the β-dominance relation. When the value of β increases, the lower approximation operator of the $OMGRSM$ based on the β-dominance relation also increases, while the upper approximation operator correspondingly decreases.

In $OMGRSM$, to find the relationships of two approximation operators about single object set and two object sets under multiple β-dominance relations, we give the following properties.

Proposition 4.2 Let $IOIS = (U, AT, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\forall \Psi, \Upsilon \subseteq U$, the following properties in $OMGRSM$ under β-dominance relation are established:

(1) $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cap \Upsilon) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \cap \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon)$,

(2) $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cup \Upsilon) \supseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \cup \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon)$,

(3) $\Psi \subseteq \Upsilon \Rightarrow \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon)$,

(4) $\Psi \subseteq \Upsilon \Rightarrow \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \subseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon)$,

(5) $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \cup \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cup \Upsilon)$,

(6) $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cap \Upsilon) \subseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \cap \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon)$.

Proof. (1) For any $\forall \psi \in \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cap \Upsilon) \Rightarrow \exists i \in \{1, 2, \cdots, s\}$, $[\psi]_{A_i}^{\geq * \beta} \subseteq \Psi \cap \Upsilon \Rightarrow \exists i \in \{1, 2, \cdots, s\}$, $[\Psi]_{A_i}^{\geq * \beta} \subseteq \psi \wedge [\psi]_{A_i}^{\geq * \beta} \subseteq \Upsilon \Rightarrow \psi \in \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \cap \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon)$. So $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cap \Upsilon) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \cap \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon)$.

(2) For any $\forall \psi \in \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \cup \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon) \Rightarrow$ for $\forall i \in \{1, 2, \cdots, s\}$, $([\psi]_{A_i}^{\geq * \beta} \cap \Psi \neq \emptyset) \vee ([\psi]_{A_i}^{\geq * \beta} \cap \Upsilon \neq \emptyset) \Rightarrow$ for any $\forall i \in \{1, 2, \cdots, s\}$, $[\psi]_{A_i}^{\geq * \beta} \cap (\Psi \cup \Upsilon) \neq \emptyset \Rightarrow \psi \in \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cup \Upsilon)$. So $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \cup \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon) \subseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cup \Upsilon)$.

(3) Assume $\Psi \subseteq \Upsilon$, and it can obviously be known that $\Psi \cap \Upsilon = \Psi$, so $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cap \Upsilon) = \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$. From(1), we can get that $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cap \Upsilon) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \cap \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon)$, So $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon)$.

(4) Assume $\Psi \subseteq \Upsilon$, and it can obviously be known that $\Psi \cup \Upsilon = \Upsilon$, so $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cup \Upsilon) = \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon)$. From(2), we can get that $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \cup \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon) \subseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cup \Upsilon)$, So $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \subseteq$

$\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon)$.

(5) Because $\Psi \subseteq \Psi \cup \Upsilon$ and $\Upsilon \subseteq \Psi \cup \Upsilon$, From(3), so it can be got that $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cup \Upsilon)$ and $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cup \Upsilon)$. Hence $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \cup \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cup \Upsilon)$.

(6) Because $\Psi \cap \Upsilon \subseteq \Psi$ and $\Psi \cap \Upsilon \subseteq \Upsilon$, From(4), so it can be got that $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cap \Upsilon) \subseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$ and $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cap \Upsilon) \subseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon)$. Hence $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi \cap \Upsilon) \subseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \cap \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Upsilon)$.

The proposition was proved.

proposition 4.2 indicates that as the object set expands, both the upper and lower approximation operators of the $OMGRSM$ based on the β-dominance relation exhibit an increasing trend.

4.2 Pessimistic multiple granulation rough set model

In this subsection, similar to the $OMGRSM$, we propose a pessimistic multiple granulation rough set model based on β-dominance relation in the $IOIS$ and investigate its related properties.

Definition 4.2 Let $IOIS = (U, AT, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\forall \Psi \subseteq U$. The lower and upper approximation operators of Ψ for pessimistic multiple granulation of β-dominance relation under attribute sets $A_1, A_2, ..., A_s$ are respectively defined as:

$$\underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi) = \{\psi \in U | \wedge_{i=1}^{s} [\psi]_{A_i}^{\geq * \beta} \subseteq \Psi\}, \ \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi) = \{\psi \in U | \vee_{i=1}^{s} [\psi]_{A_i}^{\geq * \beta} \cap \Psi \neq \emptyset\}.$$

$[\underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi), \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi)]$ is called $PMGRSM$ under β-dominance relation in the multiple granulation spaces $A_1, A_2, \cdots, A_s$. Moreover, if $\underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi) = \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi)$, then Ψ is a definable set about multiple granulation spaces $A_1, A_2, \cdots, A_s$. If not, Ψ is a rough set in the $IOIS$.

The uncertainty or boundary area of Ψ under $PMGRSM$ of β-dominance relation is defined by using the following method.

$$\sum_{i=1}^{s} BND_{i\beta}^{P}(\Psi) = \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi) - \underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi).$$

It can be known from definition 4.2 that the two approximation operators for lower and upper approximations in $PMGRSM$ are obtained by utilizing the object dominance classes derived from multiple β-dominance relations. If all granulation spaces are equal ($A_1 = A_2 = \cdots = A_s$), then $PMGRSM$ under β-dominance relation will become single granulation rough set model under β-dominance relation.

According to definition 4.2, the following properties of $PMGRSM$ under β-dominance relation are got by us in the $IOIS$.

Proposition 4.3 Let $IOIS = (U, AT, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\forall \Psi \subseteq U$, then the following properties in $PMGRSM$ under β-dominance relation are established:

(1) $\underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi) \subseteq \Psi \subseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi)$,

(2) $\underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\emptyset) = \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\emptyset) = \emptyset$, $\underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(U) = \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(U) = U$,

(3) $\underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\sim \Psi) = \sim \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi), \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\sim \Psi) = \sim \underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi)$,

(4) $\beta_1 \leq \beta_2 \Rightarrow \underline{\sum_{i=1}^{s} A_{i\beta_1}^{P}}(\Psi) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta_2}^{P}}(\Psi)$,

(5) $\beta_1 \leq \beta_2 \Rightarrow \overline{\sum_{i=1}^{s} A_{i\beta_2}^{P}}(\Psi) \subseteq \overline{\sum_{i=1}^{s} A_{i\beta_1}^{P}}(\Psi)$.

Proof. Combining definition 4.2, it can be easily proven by following the proof approach of proposition 4.1.

Proposition 4.3 demonstrates the correlation between the upper and lower approximation operators and the given object set in the $PMGRSM$ based on the β-dominance relation. When the value of

β increases, the lower approximation operator of the $PMGRSM$ based on the β-dominance relation also increases, while the upper approximation operator correspondingly decreases.

In the $PMGRSM$, to find the relationships of two approximation operators about single object set and two object sets under multiple β-dominance relations, we give the following properties.

Proposition 4.4 Let $IOIS = (U, AT, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\forall \Psi, \Upsilon \subseteq U$, the following properties in the $PMGRSM$ under β-dominance relation are established:

(1) $\underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi \cap \Upsilon) = \underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi) \cap \underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Upsilon)$,

(2) $\overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi \cup \Upsilon) = \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi) \cup \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Upsilon)$,

(3) $\Psi \subseteq \Upsilon \Rightarrow \underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Upsilon)$,

(4) $\Psi \subseteq \Upsilon \Rightarrow \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi) \subseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Upsilon)$,

(5) $\underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi) \cup \underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Upsilon) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi \cup \Upsilon)$,

(6) $\overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi \cap \Upsilon) \subseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi) \cap \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Upsilon)$.

Proof. Combining definition 4.2, it can be easily proven by following the proof approach of proposition 4.2.

proposition 4.4 indicates that as the object set expands, both the upper and lower approximation operators of the $PMGRSM$ based on the β-dominance relation exhibit an increasing trend.

4.3 Accuracy measure and approximation quality

As we all know, the uncertainty of knowledge is determined by the ambiguity in the border region. In the multiple granulation rough set model, the larger the boundary area of the set, the smaller its accuracy measure. In this subsection, we present some important measures in two types of multiple granulation rough set models under the β-dominance relation, including accuracy measure, approximation quality, and also design their corresponding algorithms.

Definition 4.3 Let $IOIS = (U, AT, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\forall \Psi \subseteq U$. The accuracy measures of $OMGRSM$ and $PMGRSM$ based on β-dominance relation with respect to Ψ under $A_1, A_2, ..., A_s$ are respectively defined as:

$$\alpha_{\beta}^{O}(\sum_{i=1}^{s} A_i, \Psi) = \frac{|\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)|}{|\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)|},\ \alpha_{\beta}^{P}(\sum_{i=1}^{s} A_i, \Psi) = \frac{|\underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi)|}{|\overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi)|},$$

where $\Psi \neq \emptyset$.

To facilitate the calculation of accuracy measure, we propose an accuracy measure algorithm based on the $OMGRSM$ under the β-dominance relation. The detailed steps are provided in Algorithm 2, and its time complexity is given as follows. Step 4 calculates the time complexity of all objects' β-dominance classes as $O(|AT||U|^2)$(see Algorithm 1). Steps 5 to 10 demonstrate that the time complexity for computing both the upper approximation operator and lower approximation operator of $OMGRSM$ based on β-dominance relation under attribute set is $O(|U|^3)$. Steps 3 to 11 demonstrate that the time complexity for computing the upper approximation operator and lower approximation operator of $OMGRSM$ based on β-dominance relation under multiple attribute sets is $O(2^{|AT|}|AT||U|^2{+}2^{|AT-1|}|U|^3)$. Therefore, the time complexity of Algorithm 2 is $O(2^{|AT|}|AT||U|^2{+}2^{|AT-1|}|U|^3)$.

Regarding the accuracy measure algorithm of the $PMGRSM$ under the β-dominance relation, by incorporating the concept of accuracy measure as defined in definition 4.3 within the $PMGRSM$ under the β-dominance relation, it becomes relatively straightforward to design with reference to Algorithm 2.

Proposition 4.5 Let $IOIS = (U, AT, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\forall \Psi \subseteq U$, some properties of $OMGRSM$ and $PMGRSM$ under β-dominance relation are established:

(1) $\beta_1 \leq \beta_2 \Rightarrow \alpha_{\beta_1}^{O}(\sum_{i=1}^{s} A_i, \Psi) \leq \alpha_{\beta_2}^{O}(\sum_{i=1}^{s} A_i, \Psi)$,

Algorithm 2: An algorithm for computing the accuracy measure of $OMGRSM$ based on β-dominance relation in the $IOIS$

Input : An $IOIS = (U, AT, V, f)$, $A_i \in AT(i = 1, ..., s)$,$\Psi \subseteq U$
Output : $\alpha_\beta^O(\sum_{i=1}^s A_i, \Psi)$
begin
  Initialize $\underline{\sum_{i=1}^s A_{i\beta}^O}(\Psi) \leftarrow \emptyset, \overline{\sum_{i=1}^s A_{i\beta}^O}(\Psi) \leftarrow \emptyset$;
  for $i = 1$ *to* s **do**
    Return to Algorithm 1 and calculate each $\psi_j \in U$, $[\psi_j]_{A_i}^{\geq *\beta}$;
    if $[\psi_j]_{A_i}^{\geq *\beta} \subseteq \Psi$ **then**
      $\underline{\sum_{i=1}^s A_{i\beta}^O}(\Psi) \leftarrow \underline{\sum_{i=1}^s A_{i\beta}^O}(\Psi) \cup \{\psi_j\}$;
    if *each* $[\psi_j]_{A_i}^{\geq *\beta} \bigcap \Psi \neq \emptyset$ **then**
      $\overline{\sum_{i=1}^s A_{i\beta}^O}(\Psi) \leftarrow \overline{\sum_{i=1}^s A_{i\beta}^O}(\Psi) \cup \{\psi_j\}$;
  Compute $\alpha_\beta^O(\sum_{i=1}^s A_i, \Psi) = |\underline{\sum_{i=1}^s A_{i\beta}^O}(\Psi)|/|\overline{\sum_{i=1}^s A_{i\beta}^O}(\Psi)|$.
return: $\alpha_\beta^O(\sum_{i=1}^s A_i, \Psi)$.

(2) $\beta_1 \leq \beta_2 \Rightarrow \alpha_{\beta_1}^P(\sum_{i=1}^s A_i, \Psi) \leq \alpha_{\beta_2}^P(\sum_{i=1}^s A_i, \Psi)$.

Proof. (1) For any $\forall \psi \in U$,$\forall A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\beta_1 \leq \beta_2$, $\forall \Psi \subseteq U$, by proposition 4.1, we can get $\underline{\sum_{i=1}^s A_{i\beta_1}^O}(\Psi) \subseteq \underline{\sum_{i=1}^s A_{i\beta_2}^O}(\Psi)$ and $\overline{\sum_{i=1}^s A_{i\beta_2}^O}(\Psi) \subseteq \overline{\sum_{i=1}^s A_{i\beta_1}^O}(\Psi)$. Combine with Definition 4.3, we can have $\alpha_{\beta_1}^O(\sum_{i=1}^s A_i, \Psi) \leq \alpha_{\beta_2}^O(\sum_{i=1}^s A_i, \Psi)$.

(2) For any $\forall \psi \in U$, $\forall A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\beta_1 \leq \beta_2$, $\forall \Psi \subseteq U$, by proposition 4.3, we can get $\underline{\sum_{i=1}^s A_{i\beta_1}^P}(\Psi) \subseteq \underline{\sum_{i=1}^s A_{i\beta_2}^P}(\Psi)$ and $\overline{\sum_{i=1}^s A_{i\beta_2}^P}(\Psi) \subseteq \overline{\sum_{i=1}^s A_{i\beta_1}^P}(\Psi)$. Combine with Definition 4.3, we can have $\alpha_{\beta_1}^P(\sum_{i=1}^s A_i, \Psi) \leq \alpha_{\beta_2}^P(\sum_{i=1}^s A_i, \Psi)$.

The proposition was proved.

Proposition 4.5 indicates that as the value of β increases, the accuracy measures of both the $OMGRSM$ and $PMGRSM$ under β-dominance relation increase.

Definition 4.4 Let $IODS = (U, AT \cup d, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\{D_1, D_2, \cdots, D_k\}$ is partition of decision class, which is described by d. The approximation qualities of $OMGRSM$ and $PMGRSM$ based on the β-dominance relation are defined with respect to the partition of d under $A_1, A_2, ..., A_s$ as follows:

$$\gamma_\beta^O(\sum_{i=1}^s A_i, d) = \frac{\sum_{l=1}^k(|\underline{\sum_{i=1}^s A_{i\beta}^O}(D_l)|)}{|U|},\ \gamma_\beta^P(\sum_{i=1}^s A_i, d) = \frac{\sum_{l=1}^k(|\underline{\sum_{i=1}^s A_{i\beta}^P}(D_l)|)}{|U|},$$

where $R_d = \{(\psi_1, \psi_2) \in U \times U | g(\psi_1, d) = g(\psi_2, d)\}$, $U/d = \{[\psi]_d, \forall \psi \in U\} = \{D_1, D_2, \cdots, D_k\}$, $\gamma_\beta^O(\sum_{i=1}^s A_i, d)$ and $\gamma_\beta^P(\sum_{i=1}^s A_i, d)$ are respectively elaborate on the percentage of objects whose knowledge obtained through the divisions of $A_1, A_2, ..., A_s$ can be accurately classified into decision classes.

Given that the approximation quality algorithm design for the $OMGRSM$ under β-dominance relation is similar to that of the $PMGRSM$ under β-dominance relation, this paper only elaborates on the approximation quality algorithm for the $OMGRSM$ under β-dominance relation. For detailed steps, please refer to Algorithm 3, whose time complexity be specifically explained as follows.

Step 4 calculates the time complexity of all objects' β-dominance classes as $O(|AT||U|^2)$(see Algorithm 1), and calculate the time complexity of each object's decision class under the decision attribute d as $O(|U|^2)$. Steps 5 to 7 demonstrate that the time complexity for computing the lower approximation operator of $OMGRSM$ based on β-dominance relation under attribute set is $O(|U|^3)$. Steps 3 to 8 demonstrate that the time complexity for computing the lower approximation operator of $OMGRSM$ based on β-dominance relation under multiple attribute sets is $O(2^{|AT-1|}|AT||U|^2{+}2^{|AT-1|}|U|^3)$.

Therefore, the time complexity of Algorithm 3 is $O(2^{|AT-1|}|AT||U|^2{+}2^{|AT-1|}|U|^3)$.

Algorithm 3: An algorithm for computing the approximation quality of $OMGRSM$ based on β-dominance relation in the $IODS$

Input : An $IODS = (U, AT \bigcup d, V, f)$, $A_i \in AT(i = 1, ..., s)$;
Output : $\gamma_\beta^O(\sum_{i=1}^s A_i, d)$.
begin
  Initialize $D_l = [\psi_j]_d \leftarrow \emptyset(l = 1, 2, .., k)$, $\underline{\sum_{i=1}^s A_{i\beta}^O}(D_l) \leftarrow \emptyset$.
  for $i = 1$ *to* s **do**
    Return to Algorithm 1 and calculate each $\psi_j \in U$, $[\psi_j]_{A_i}^{\geq * \beta}$, meanwhile, calculate $D_l = [\psi_j]_d = \{\forall \psi_h \in U | g(\psi_h, d) = g(\psi_j, d)\} \bigcup \{\psi_j\}$;
    if $[\psi_j]_{A_i}^{\geq * \beta} \subseteq D_l$ **then**
      $\underline{\sum_{i=1}^s A_{i\beta}^O}(D_l) \leftarrow \underline{\sum_{i=1}^s A_{i\beta}^O}(D_l) \cup \{\psi_j\}$;
  Compute $\gamma_\beta^O(\sum_{i=1}^s A_i, d) = \frac{\sum_{l=1}^k (|\underline{\sum_{i=1}^s A_{i\beta}^O}(D_l)|)}{|U|}$.
return: $\gamma_\beta^O(\sum_{i=1}^s A_i, d)$.

Proposition 4.6 Let $IODS = (U, AT \cup d, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\{D_1, D_2, \cdots, D_k\}$ is all decision classes, $\forall \Psi \subseteq U$, some properties of $OMGRSM$ and $PMGRSM$ under β-dominance relation are established:

(1) $\beta_1 \leq \beta_2 \Rightarrow \gamma_{\beta_1}^O(\sum_{i=1}^s A_i, d) \leq \gamma_{\beta_2}^O(\sum_{i=1}^s A_i, d)$.

(2) $\beta_1 \leq \beta_2 \Rightarrow \gamma_{\beta_1}^P(\sum_{i=1}^s A_i, d) \leq \gamma_{\beta_2}^P(\sum_{i=1}^s A_i, d)$.

Proof.

(1) For any $\forall \Psi \subseteq U$, $\beta_1 \leq \beta_2$, Combine with proposition 4.1, we can get $\underline{\sum_{i=1}^s A_{i\beta_1}^O}(\Psi) \subseteq \underline{\sum_{i=1}^s A_{i\beta_2}^O}(\Psi)$. Due to $\{D_1, D_2, \cdots, D_k\}$ is all decision classes of the object, for any $\forall j = 1, 2, \cdots, k$, $\underline{\sum_{i=1}^s A_{i\beta_1}^O}(D_j) \subseteq \underline{\sum_{i=1}^s A_{i\beta_2}^O}(D_j)$, i.e, $\sum_{j=1}^k (\underline{\sum_{i=1}^s A_{i\beta_1}^O}(D_j)) \subseteq \sum_{j=1}^k (\underline{\sum_{i=1}^s A_{i\beta_2}^O}(D_j))$. Hence, Combine with Definition 4.4, we can have $\gamma_{\beta_1}^O(\sum_{i=1}^s A_i, d) \leq \gamma_{\beta_2}^O(\sum_{i=1}^s A_i, d)$.

(2) For any $\forall \Psi \subseteq U$, $\beta_1 \leq \beta_2$, by proposition 4.3, we can get $\underline{\sum_{i=1}^s A_{i\beta_1}^P}(\Psi) \subseteq \underline{\sum_{i=1}^s A_{i\beta_2}^P}(\Psi)$. Due to $\{D_1, D_2, \cdots, D_k\}$ is all decision classes induced of the object, for any $\forall j = 1, 2, \cdots, k$, $\underline{\sum_{i=1}^s A_{i\beta_1}^P}(D_j) \subseteq \underline{\sum_{i=1}^s A_{i\beta_2}^P}(D_j)$, i.e, $\sum_{j=1}^k (\underline{\sum_{i=1}^s A_{i\beta_1}^P}(D_j)) \subseteq \sum_{j=1}^k (\underline{\sum_{i=1}^s A_{i\beta_2}^P}(D_j))$. Hence, combine with Definition 4.4, we can have $\gamma_{\beta_1}^P(\sum_{i=1}^s A_i, d) \leq \gamma_{\beta_2}^P(\sum_{i=1}^s A_i, d)$.

The proposition was proved.

Proposition 4.6 indicates that as the value of β increases, the approximation qualities of both the $OMGRSM$ and $PMGRSM$ under β-dominance relation increase.

In order to conveniently calculate the accuracy measures and approximation qualities of the $OMGRSM$ and $PMGRSM$ under β-dominance relation with different thresholds , it is necessary to give the following properties.

Proposition 4.7 Let $IOIS = (U, AT, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\forall \Psi \subseteq U$, then:

(1) $\underline{\sum_{i=1}^s A_{i\beta}^P}(\Psi) \subseteq \underline{\sum_{i=1}^s A_{i\beta}^O}(\Psi)$,

(2) $\overline{\sum_{i=1}^s A_{i\beta}^O}(\Psi) \subseteq \overline{\sum_{i=1}^s A_{i\beta}^P}(\Psi)$.

Proof. (1) For any $\forall \psi \in \underline{\sum_{i=1}^s A_{i\beta}^P}(\Psi)$, by definition 4.2, we know for any $\forall i \in \{1, 2, \cdots, s\}$, $[\psi]_{A_i}^{\geq * \beta} \subseteq \Psi$. And we get $\psi \in \{\psi \in U | \vee_{i=1}^s [\psi]_{A_i}^{\geq * \beta} \subseteq \Psi\}$. Therefore, $\psi \in \underline{\sum_{i=1}^s A_{i\beta}^O}(\Psi)$, i.e, $\underline{\sum_{i=1}^s A_{i\beta}^P}(\Psi) \subseteq \underline{\sum_{i=1}^s A_{i\beta}^O}(\Psi)$.

(2) For any $\forall \psi \in \overline{\sum_{i=1}^s A_{i\beta}^O}(\Psi)$, by definition 4.1, we know for any $\forall i \in \{1, 2, \cdots, s\}$, $[\psi]_{A_i}^{\geq * \beta} \cap \Psi \neq \emptyset$. And we get $\psi \in \{\psi \in U | \vee_{i=1}^s [\psi]_{A_i}^{\geq * \beta} \cap \Psi \neq \emptyset\}$. Therefore, $\psi \in \overline{\sum_{i=1}^s A_{i\beta}^P}(\Psi)$, i.e, $\overline{\sum_{i=1}^s A_{i\beta}^O}(\Psi) \subseteq \overline{\sum_{i=1}^s A_{i\beta}^P}(\Psi)$.

The proposition was proved.

Proposition 4.8 Let $IOIS = (U, AT, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\forall \Psi \subseteq U$, $0 < \beta \leq 1$, $0 < \beta_1 \leq 1$ and $0 < \beta_2 \leq 1$, then:

(1) If $\underline{\sum_{i=1}^{s} A_{i\beta_1}^{O}}(\Psi) = \Psi$, when $\beta_1 \leq \beta_2 \leq 1$, then $\underline{\sum_{i=1}^{s} A_{i\beta_2}^{O}}(\Psi) = \Psi$;

(2) If $\underline{\sum_{i=1}^{s} A_{i\beta_1}^{P}}(\Psi) = \Psi$, when $\beta_1 \leq \beta_2 \leq 1$, then $\underline{\sum_{i=1}^{s} A_{i\beta_2}^{P}}(\Psi) = \Psi$;

(3) If $\overline{\sum_{i=1}^{s} A_{i\beta_2}^{O}}(\Psi) = U$, when $0 < \beta_1 \leq \beta_2$, then $\overline{\sum_{i=1}^{s} A_{i\beta_1}^{O}}(\Psi) = U$;

(4) If $\overline{\sum_{i=1}^{s} A_{i\beta_2}^{P}}(\Psi) = U$, when $0 < \beta_1 \leq \beta_2$, then $\overline{\sum_{i=1}^{s} A_{i\beta_1}^{P}}(\Psi) = U$;

(5) If $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) = \emptyset$, then $\underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi) = \emptyset$;

(6) If $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) = U$, then $\overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi) = U$.

Proof. Combine with proposition 4.1, proposition 4.3 and proposition 4.7, it can be obtained easily.

The propositions 4.7 and 4.8 illustrate the differences and relationships between the two models, $OMGRSM$ and $PMGRSM$, under the β-dominance relation. They also demonstrated that under the β-dominance relation, the lower approximation of the $OMGRSM$ contains the lower approximation of the $PMGRSM$, whereas the upper approximation of the $OMGRSM$ is contained within the upper approximation of the $PMGRSM$ under the β-dominance relation.

5 Comparative analysis of three multiple granulation rough set models

In this section, we will discuss the relationships between upper and lower approximation operators, accuracy measures, and approximation qualities in multiple granulation rough set models based on β-dominance relation, EDR and LDR in the $IOIS$.

Proposition 5.1 Let $IOIS = (U, AT, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\forall \Psi \subseteq U$, then:

(1) $\underline{\sum_{i=1}^{s} A_{iE}^{O}}(\Psi) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \subseteq \underline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi)$,

(2) $\overline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi) \subseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \subseteq \overline{\sum_{i=1}^{s} A_{iE}^{O}}(\Psi)$,

(3) $\sum_{i=1}^{s} BND_{iL}^{O}(\Psi) \subseteq \sum_{i=1}^{s} BND_{i\beta}^{O}(\Psi) \subseteq \sum_{i=1}^{s} BND_{iE}^{O}(\Psi)$.

Proof. (1a) For any $\forall \psi \in \underline{\sum_{i=1}^{s} A_{iE}^{O}}(\Psi)$, by Definition 2.3, we obtain $\exists i \in \{1, 2, \cdots, s\}$, makes $[\psi]_{A_i}^{\geq *E} \subseteq \Psi$. Since $\psi \in U$, for any $\forall i \in \{1, 2, \cdots, s\}$, makes $[\psi]_{A_i}^{\geq *\beta} \subseteq [\psi]_{A_i}^{\geq *E}$. Therefore, exists $\exists i \in \{1, 2, \cdots, s\}$, makes $[\psi]_{A_i}^{\geq *\beta} \subseteq \Psi$, we can get $\psi \in \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$. Hence, $\underline{\sum_{i=1}^{s} A_{iE}^{O}}(\Psi) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$.

(1b) For any $\forall \psi \in \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$, by Definition 4.1, we obtain $\exists i \in \{1, 2, \cdots, s\}$, makes $[\psi]_{A_i}^{\geq *\beta} \subseteq \Psi$. Since $\psi \in U$, $\forall i \in \{1, 2, \cdots, s\}$, and $[\psi]_{A_i}^{\geq *L} \subseteq [\psi]_{A_i}^{\geq *\beta}$. Therefore, exists $\exists i \in \{1, 2, \cdots, s\}$, makes $[\psi]_{A_i}^{\geq *L} \subseteq \Psi$, we can get $\psi \in \underline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi)$. Hence, $\underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \subseteq \underline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi)$.

(2a) For any $\forall \psi \in \overline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi)$, by Definition 2.3, we obtain $\forall i \in \{1, 2, \cdots, s\}, [\psi]_{A_i}^{\geq *L} \cap \psi \neq \emptyset$. Therefore, we get $\psi \in \{\psi \in U | \wedge_{i=1}^{s} [\psi]_{A_i}^{\geq *L} \cap \Psi \neq \emptyset\}$. Since $\psi \in U$, for any $\forall i \in \{1, 2, \cdots, s\}$, makes $[\psi]_{A_i}^{\geq *L} \subseteq [\psi]_{A_i}^{\geq *\beta}$. Therefore, for any $\forall i \in \{1, 2, \cdots, s\}$, makes $[\psi]_{A_i}^{\geq *\beta} \cap \Psi \neq \emptyset$, we can get $\psi \in \{\psi \in U | \wedge_{i=1}^{s} [\psi]_{A_i}^{\geq *\beta} \cap \Psi \neq \emptyset\}$. Hence, $\overline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi) \subseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$.

(2b) For any $\forall \psi \in \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$, by Definition 4.1, we obtain $\forall i \in \{1, 2, \cdots, s\}, [\psi]_{A_i}^{\geq *\beta} \cap \Psi \neq \emptyset$. Therefore, we get $\psi \in \{\psi \in U | \wedge_{i=1}^{s} [\psi]_{A_i}^{\geq *\beta} \cap \Psi \neq \emptyset\}$. Since $\psi \in U$, for any $\forall i \in \{1, 2, \cdots, s\}$, makes $[\psi]_{A_i}^{\geq *\beta} \subseteq [\psi]_{A_i}^{\geq *E}$. Therefore, for any $\forall i \in \{1, 2, \cdots, s\}$, makes $[\psi]_{A_i}^{\geq *E} \cap \Psi \neq \emptyset$, we can get $\psi \in \{\psi \in U | \wedge_{i=1}^{s} [\psi]_{A_i}^{\geq *E} \cap \Psi \neq \emptyset\}$. Hence, $\overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) \subseteq \overline{\sum_{i=1}^{s} A_{iE}^{O}}(\Psi)$.

(3a) Because $\sum_{i=1}^{s} BND_{iL}^{O}(\Psi) = \overline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi) - \underline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi), \sum_{i=1}^{s} BND_{i\beta}^{O}(\Psi) = \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) - \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$. Applying the results of (1b) and (2a), so we have that $\sum_{i=1}^{s} BND_{iL}^{O}(\Psi) \subseteq \sum_{i=1}^{s} BND_{i\beta}^{O}(\Psi)$.

(3b) Because $\sum_{i=1}^{s} BND_{iE}^{O}(\Psi) = \overline{\sum_{i=1}^{s} A_{iE}^{O}}(\Psi) - \underline{\sum_{i=1}^{s} A_{iE}^{O}}(\Psi), \sum_{i=1}^{s} BND_{i\beta}^{O}(\Psi) = \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi) - \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$. Applying the results of (1a) and (2b), so we have that $\sum_{i=1}^{s} BND_{i\beta}^{O}(\Psi) \subseteq \sum_{i=1}^{s} BND_{iE}^{O}(\Psi)$.

The proposition was proved.

Based on the above results, the following properties about $PMGRSM$ are given.

Proposition 5.2 Let $IOIS = (U, AT, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\forall \Psi \subseteq U$, then:

(1) $\underline{\sum_{i=1}^{s} A_{iE}^{P}}(\Psi) \subseteq \underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi) \subseteq \underline{\sum_{i=1}^{s} A_{iL}^{P}}(\Psi)$,

(2) $\overline{\sum_{i=1}^{s} A_{iL}^{P}}(\Psi) \subseteq \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi) \subseteq \overline{\sum_{i=1}^{s} A_{iE}^{P}}(\Psi)$,

(3) $\sum_{i=1}^{s} BND_{iL}^{P}(\Psi) \subseteq \sum_{i=1}^{s} BND_{i\beta}^{P}(\Psi) \subseteq \sum_{i=1}^{s} BND_{iE}^{P}(\Psi)$.

Proof. Similar to the proofs of proposition 5.1 can be easily got.

Proposition 5.1 and Proposition 5.2 illustrate that the lower approximation operator sets of $OMGRSM$ and $PMGRSM$ based on the β-dominance relation not only include the lower approximation operator sets of $OMGRSM$ and $PMGRSM$ based on the EDR, but are also contained within the lower approximation operator sets of $OMGRSM$ and $PMGRSM$ based on the LDR in the $IOIS$. Similarly, the upper approximation operator sets of $OMGRSM$ and $PMGRSM$ based on the β-dominance relation not only include the upper approximation operator sets of $OMGRSM$ and $PMGRSM$ based on the LDR, but are also contained within the upper approximation operator sets of $OMGRSM$ and $PMGRSM$ based on the EDR.

To further investigate the relationships among multiple granulation rough set models based on β-dominance relation, EDR, and LDR, $\forall a \in A$, suppose n indicates the maximum value of the conversion value $V_a^{'}(\psi)$. When $0 < \beta \leq 1/n$, $OMGRSM$ and $PMGRSM$ based on the β-dominance relation transform into $OMGRSM$ and $PMGRSM$ of EDR; when $\beta = 1$, $OMGRSM$ and $PMGRSM$ based on the β-dominance relation transform into $OMGRSM$ and $PMGRSM$ of LDR. That is as follows:

(1) For $0 < \beta \leq 1/n$, and we can have $\underline{\sum_{i=1}^{s} A_{iE}^{O}}(\Psi) = \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$,$\overline{\sum_{i=1}^{s} A_{iE}^{O}}(\Psi) = \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$, $\underline{\sum_{i=1}^{s} A_{iE}^{P}}(\Psi) = \underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi)$,$\overline{\sum_{i=1}^{s} A_{iE}^{P}}(\Psi) = \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi)$.

(2) For $\beta = 1$, and we can have $\underline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi) = \underline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$,$\overline{\sum_{i=1}^{s} A_{iL}^{O}}(\Psi) = \overline{\sum_{i=1}^{s} A_{i\beta}^{O}}(\Psi)$, $\underline{\sum_{i=1}^{s} A_{iL}^{P}}(\Psi) = \underline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi)$, $\overline{\sum_{i=1}^{s} A_{iL}^{P}}(\Psi) = \overline{\sum_{i=1}^{s} A_{i\beta}^{P}}(\Psi)$.

The above conclusion can also be explained through Figure 2.

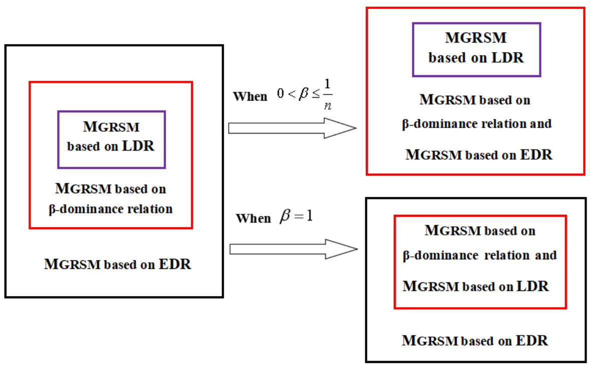

Figure 2: The relationships among $MGRSMs$ under β-dominance relation, EDR and LDR

Proposition 5.3 Let $IODS = (U, AT \cup d, V, f)$, $A \subseteq AT, A = \{A_1, A_2, \cdots, A_s\}$, $\{D_1, D_2, \cdots, D_k\}$ is partition of decision class, which is described by d, $\forall \Psi \subseteq U$, then the relationship between the accuracy measures of $OMGRSM$ and $PMGRSM$ under the three dominance relations, as well as their approximation qualities, has been clearly established as follows.

(1) $\alpha_E^O(\sum_{i=1}^{s} A_i, \Psi) \leq \alpha_\beta^O(\sum_{i=1}^{s} A_i, \Psi) \leq \alpha_L^O(\sum_{i=1}^{s} A_i, \Psi)$,

(2) $\alpha_E^P(\sum_{i=1}^{s} A_i, \Psi) \leq \alpha_\beta^P(\sum_{i=1}^{s} A_i, \Psi) \leq \alpha_L^P(\sum_{i=1}^{s} A_i, \Psi)$,

(3) $\gamma_E^O(\sum_{i=1}^{s} A_i, d) \leq \gamma_\beta^O(\sum_{i=1}^{s} A_i, d) \leq \gamma_L^O(\sum_{i=1}^{s} A_i, d)$,

(4) $\gamma_E^P(\sum_{i=1}^{s} A_i, d) \leq \gamma_\beta^P(\sum_{i=1}^{s} A_i, d) \leq \gamma_L^P(\sum_{i=1}^{s} A_i, d)$.

Proof. Combine with definition 4.3, definition 4.4, proposition 5.1 and proposition 5.2, they can be obtained easily.

Proposition 5.3 illustrates that the accuracy measures and approximation qualities of $OMGRSM$ and $PMGRSM$ based on the β-dominance relation are higher than those of $OMGRSM$ and $PMGRSM$ based on EDR, but lower than those of $OMGRSM$ and $PMGRSM$ based on LDR.

In comparison, $OMGRSM$ and $PMGRSM$ based on the β-dominance relation demonstrate superior knowledge acquisition and fault tolerance capabilities, yielding more reasonable accuracy measure and approximation quality.

6 Experimental analysis

To further illustrate the advantages of the multiple granulation rough set model under β-dominance relation in handling incomplete ordered information system, we select dataset from University of California, Irvine Machine Learning Repository (UCI, http://archive.ics.uci.edu/ml/datasets.html) and the data in Table 1 for experimentation. A comparative analysis is carried out to assess the accuracy measure and approximation quality between the multiple granulation rough set model under the β-dominance relation and those under the EDR and the LDR. Table 2 details the information of the test datasets. Owing to space constraints, we validated the proposed $OMGRSM$ using the datasets. The $PMGRSM$ is analogous to the $OMGRSM$, so it will not be elaborated on here. All experiments are implemented using PyCharm Community Edition 2021 software on a 64-bit windows system equipped with an Intel(R) Core(TM) i5-10210, 2.11 GHz processor and 8.0GB memory.

In fact, the downloaded dataset UKM is a complete dataset. Therefore, for the convenience of research, we need to randomly expand the missing values in this dataset by three percent of the total quantity. Additionally, in the multiple granulation rough set model, the required granularities are usually two or more. We assume that the granularity set of dataset ESR is $A1 = \{a_1, a_2, a_3\}$, $A2 = \{a_4, a_5\}$, with the candidate set or decision class being $\Psi = \{\psi_2, \psi_6, \psi_7, \psi_{10}\}$; the granularity set of dataset UKM is $A1 = \{a_1, a_2, a_3\}$, $A2 = \{a_4, a_5\}$. The last attribute UNS is regarded as the decision attribute d. Under the constraint of this decision attribute, objects are partitioned based on equivalence relations. We designed two candidate sets or decision classes for algorithm experiments: the first candidate set or decision class includes objects with decision attribute values of High and Middle; the second candidate set or decision class includes objects with decision attribute values of High, Middle, and Low. The computational results of Algorithm 2 and Algorithm 3 are presented as follows.

Table 1: the ranking of excellent students in an $IOIS$

Student	a_1	a_2	a_3	a_4	a_5	*Decision*
ψ_1	75	80	75	76	75	N
ψ_2	90	86	*	80	81	Y
ψ_3	82	75	85	75	*	N
ψ_4	83	76	86	*	89	N
ψ_5	86	77	83	77	85	N
ψ_6	91	85	87	76	84	Y
ψ_7	100	86	89	86	83	Y
ψ_8	90	89	*	88	82	N
ψ_9	86	*	90	89	75	N
ψ_{10}	*	100	100	100	100	Y

Table 2: The basic information of datasets

NO.	*Dataset*	*Abbreviation*	*Objects*	*Attributes*	$missing-Number$
1	$excellent-students-ranking$	ESR	10	5	6
2	$user-knowledge-modeling$	UKM	403	5	60

Figures 3, 4, and 5 depict the trends of the accuracy measures and approximation qualities in the OMGRSM based on β-dominance relation, EDR, and LDR. Following the sequence of EDR ,LDR, and β-dominance relation, as the value of β increases, the accuracy measures and the approximation qualities of three $OMGRSMs$ exhibit an ascending trend. Specifically, the $OMGRSM$ under the EDR exhibits the smallest values in terms of accuracy measure and approximation quality, indicating the poorest performance; whereas the $OMGRSM$ under the LDR achieves the largest values in accuracy measure and approximation quality, but imposes the most stringent requirements. However, as the value of β varies, the $OMGRSM$ based on the β-dominance relation can achieve accuracy measure and approximation quality that are either equal to those of the $OMGRSMs$ based on EDR and LDR, or fall within the range between their respective value intervals. This result indicates that the $OMGRSM$ based on the β-dominance relation possesses favorable fault tolerance and approximation quality.

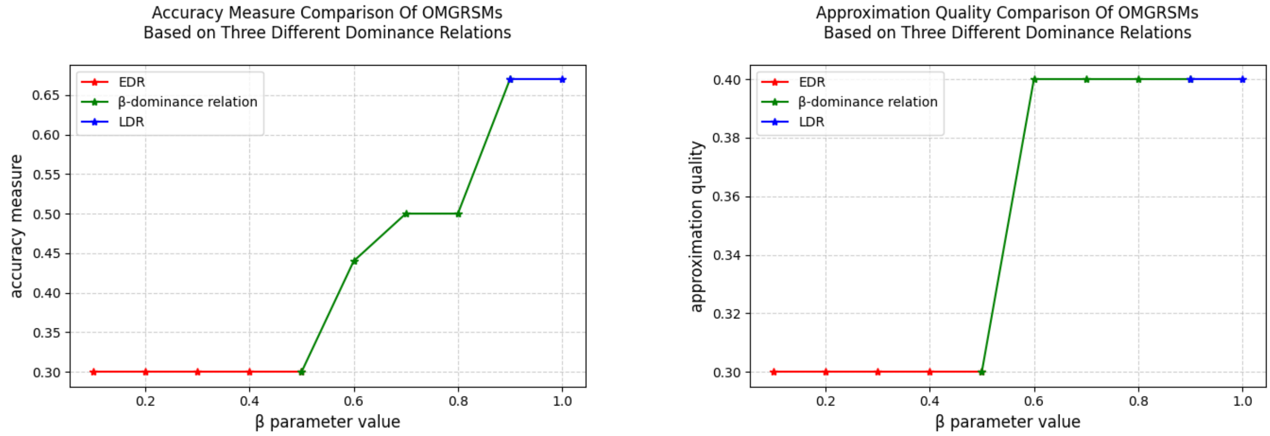

Figure 3: Experimental comparisons in ESR data

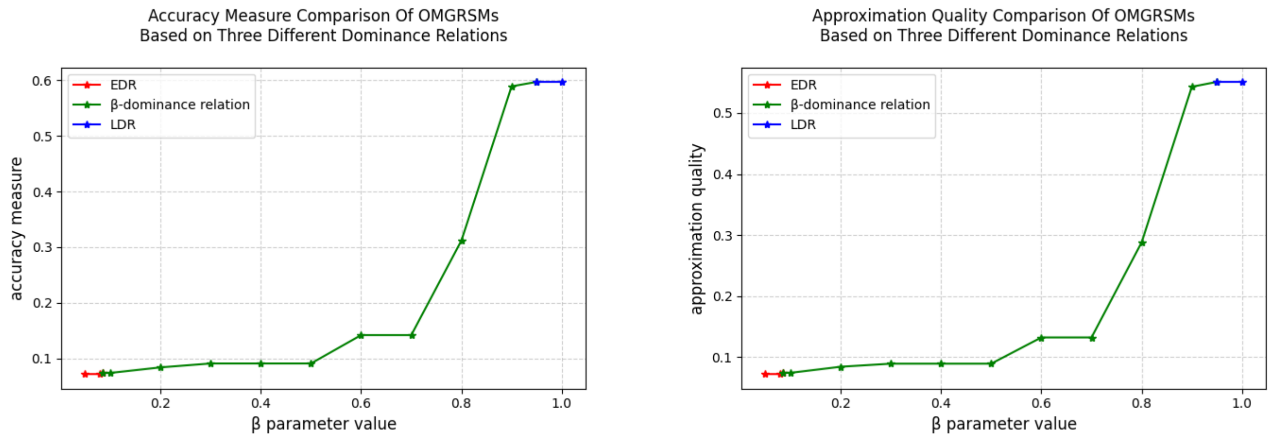

Figure 4: In the UKM dataset, experimental comparisons are conducted using decision attribute values of "high" and "medium" as candidate set

7 Conclusions

With the continuous advancement of technology, data scales are growing increasingly large, and incomplete ordered information system is widely prevalent. To enhance the fault tolerance and approximation quality of incomplete ordered information system, this paper adopts a probabilistic perspective to define the β-dominance relation for single attribute, and further proposes the concept of β-dominance relation under attribute set. Then the paper explores several properties of the β-dominance relation and provides corresponding computational algorithm. Following this, by integrating the method of granular computing, the paper combines the β-dominance relation to propose $OMGRSM$ and $PMGRSM$ based on the β-dominance relation. The properties of these models are thoroughly analyzed, and concepts such as accuracy measures and approximation qualities, along with their algorithms, are introduced. Finally, through experimental comparative analysis with multiple

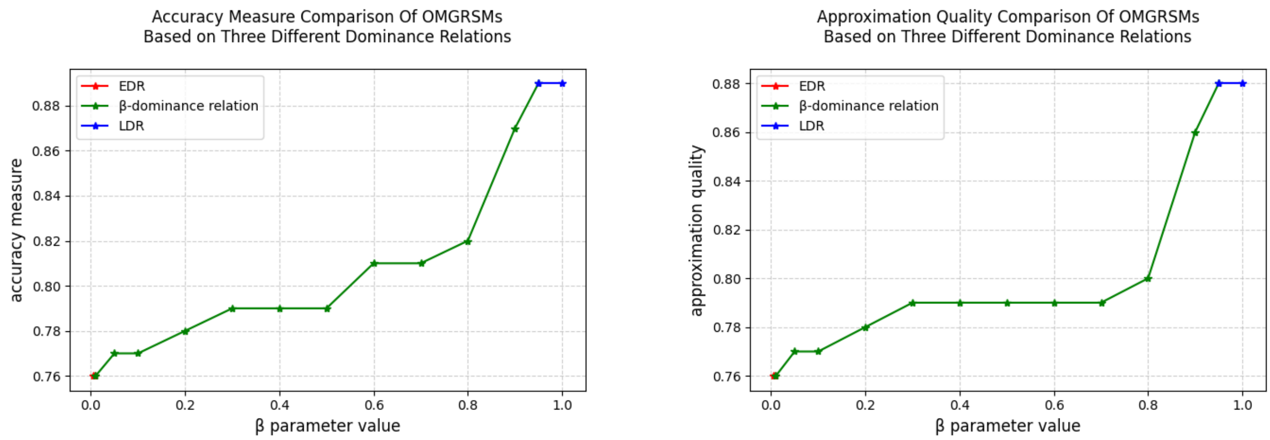

Figure 5: In the UKM dataset, experimental comparisons are conducted using the decision attribute values of "high," "medium," and "low" as candidate set

granulation rough set models based on EDR and LDR, it is found that the multiple granulation rough set model based on the β-dominance relation demonstrates significant advantages in fault tolerance and approximation quality. Although these achievements expand the application scope of incomplete ordered information system and significantly enhance their practicality, several critical issues still demand further exploration, such as feature selection and rule extraction in multiple granulation rough set model for incomplete ordered information system.

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Conflict of interest

The authors declare no conflict of interest.

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