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Approximating the Level Curves on Pascal's Surface

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Abstract

It is well-known that in general the algorithms for determining the reliability polynomial associated to a two-terminal network are computationally demanding, and even just bounding the coefficients can be taxing. Obviously, reliability polynomials can be expressed in Bernstein form, hence all the coefficients of such polynomials are fractions of the binomial coefficients. That is why we have very recently envisaged using an extension of the classical discrete *Pascal's triangle* (which comprises all the binomial coefficients) to a continuous version/surface. The fact that this continuous *Pascal's surface* has real values in between the binomial coefficients makes it appealing as being a mathematical concept encompassing *all the coefficients of all the reliability polynomials* (which are integers, as resulting from counting processes) and more. This means that, the coefficients of any reliability polynomial can be represented as discrete steps (on level curves of integer values) on *Pascal's surface*. The equation of this surface was formulated by means of the gamma function, for which quite a few approximation formulas are known. Therefore, we have started by reviewing many of those results, and have used a selection of those approximations for the level curves problem on *Pascal's surface*. Towards the end, we present fresh simulations supporting the claim that some of these could be quite useful, as being both (reasonably) easy to calculate as well as fairly accurate.

Keywords: Pascal's triangle, Pascal's surface, binomial coefficients, reliability polynomials.

1 Introduction and Motivations

The idea of starting from the well-known *discrete Pascal's triangle* and generalizing it to a *continuous Pascal's surface* is not new [32, 33, 34, 35, 47, 63, 69, 70]. Still, the novelty of this paper steams from our aim to: (*i*) use Pascal's surface for a particular application (network reliability), and (*ii*) explore level curves on Pascal's surface (see also [7]).

In computer engineering reliability was not always a hot topic. It was having its days at the dawn of computing, when the technology was mechanical [4], and yet again when transitioning to electromechanical relays and vacuum tubes [54, 55, 80]. The brisk evolution towards few nanometer transistors makes it once again prominent (see [27, 41, 43]), while fresh quantum computing developments are only adding to that [20, 42, 75]. It is in this context that we have started to revisit results on network reliability [54, 55], by looking into consecutive [5, 23] and hammock networks [22]. That is how we slowly realized that the original goal of Moore and Shannon—of finding networks for enhancing the reliability of computations—has been slowly shifting over time towards networks for reliable communications [6]. Still, with scaling, both computations and communications are being affected by variations [8], so a deeper understanding of reliability polynomials should in the end prove beneficial, and Pascal's surface seems to be providing a different view which starts from their coefficients.

The paper is structured as follows. Section 2 will show how one could extend the discrete Pascal's triangle to a continuous surface. In Section 3 we will go over quite a few approximations of the gamma function, mentioning improvements advanced over the years. Further, we shall rely on some of these approximations for estimating the binomial coefficients, while also mentioning a few direct approximations. Finally, Section 4 will introduce the Bernstein form of the reliability polynomials and bring all the pieces together by presenting preliminary simulation results. Conclusions and further directions for research are going to end the paper.

2 State-of-the-Art

2.1 Pascal's Triangle

A well-known triangular arrangement of integers is made of binomial coefficients. This is Pascal's triangle which has long been recognized from India (Pingala ca. 2nd century BC, and Halayudha ca. 10th century) to Persia (Al-Karajīca. 953-1029, Omar Khayyam 1048-1131, and Nasir al-Din al-Tusi 1201-1274), and China (Jia Xian ca. 1010-1070, Yang Hui ca. 1238-1298, and Zhu Shijie 1249-1314), as well as Europe, where it was mentioned and/or drawn in many earlier publications (see [81]):

1. Jordanus de Nemore around 1225 has drawn it on page 80 (pages are hand counted) in his "De elementis arithmetice artis" [14];
2. Petrus Apianus in 1527 has drawn it on the cover of his "Eyn newe vnnd wolgegründte underweysung aller Kauffmannß Rechnung in dreyen Büchern" [3];
3. Michael Stifel in 1544 ("Arithmetica Integra" [73]);
4. Niccolò Tartaglia in 1556 ("General Trattato di Numeri et Misure" [76]);
5. Gerolamo Cardano in 1570 ("Opus Novum de Proportionibus Numerorum" [15]); and
6. Marin Mersenne in 1636 as "Table de variétés d'un chant de 12 notes prises en 36" (Harmonie Universelle, Book 2: Des Chants, [51]), as well as in "Harmonicorum Libri XII," published also in 1636 [52].

It is extremely likely that Blaise Pascal has familiarized himself with a particular arrangement of the binomial coefficients while visiting Mersenne, both [51] and [52] being published in 1636, where the binomial coefficients were shown arranged in a rectangle. Pascal wrote his *Traité du Triangle Arithmétique* in 1654 [61], which was in fact printed only a decade later, in 1665. The following figure reproduces the triangle in Pascal's original writing (highlighting **1**, **5**, **10**, **10**, **5**, **1**).

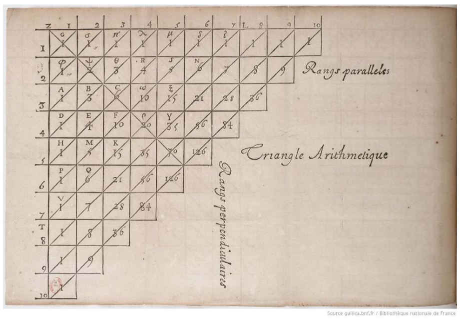


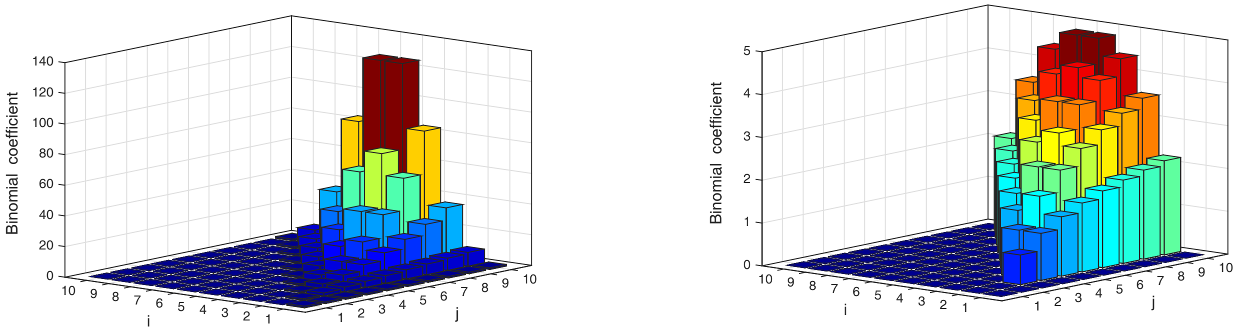
Figure 1: The triangle in Pascal's own writing (available from the National Library of France, https://gallica.bnf.fr/ark:/12148/btv1b86262012/f1.image, accessed on 06.06.2022).

As far as we know, it was Pierre Raymond de Montmort who called this arrangement *"Table de M. Pascal pour les combinaisons"* in 1708, while around 1730 Abraham de Moivre named it *"Triangulum Arithmeticum Pascalianum."* This is how most of us call it today, but it is common in Italy to sometimes call it Tartaglia's triangle, while still being called Khayyam's triangle in Iran, and Yang Hui's triangle in China. For details, the interested reader should consult not only [61], but also [10, 19, 31, 38, 49, 79].

The discrete Pascal's triangle has seen several embodiments, probably starting with the staircase of Mountain Meru (Meru-prastāra). One common representation is $L(\infty)$, which is an infinite lower triangular matrix:

$$L(\infty) = \begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ \mathbf{1} & \mathbf{5} & \mathbf{10} & \mathbf{10} & \mathbf{5} & \mathbf{1} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1}$$

where $L(\infty)_{i,j} = \binom{i}{j}$, with $\binom{i}{j} = 0$ when $i < j$. This can be seen in Fig. 2.



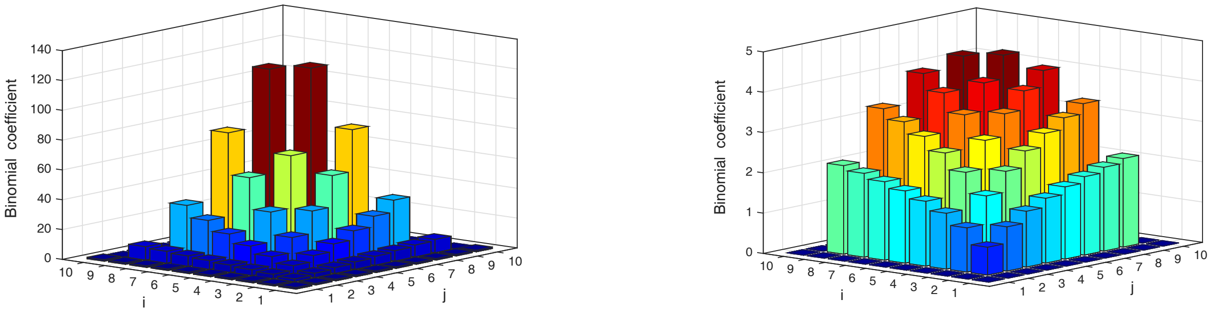
(a) (b)

Figure 2: Discrete Pascal's triangle $L(\infty)$ represented graphically as 3D bars: (a) linear scale; (b) logarithmic scale.

Another well-known form is $P(\infty)$, which is an infinite symmetric matrix:

$$P(\infty) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \mathbf{1} & \cdots \\ 1 & 2 & 3 & 4 & \mathbf{5} & 6 & \cdots \\ 1 & 3 & 6 & \mathbf{10} & 15 & 21 & \cdots \\ 1 & 4 & \mathbf{10} & 20 & 35 & 56 & \cdots \\ 1 & \mathbf{5} & 15 & 35 & 70 & 126 & \cdots \\ \mathbf{1} & 6 & 21 & 56 & 126 & 252 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2}$$

where $P(\infty)_{i,j} = \binom{i+j}{i}$, for $i, j \geq 1$ (see Fig. 3), and satisfying $P(\infty) = L(\infty) \cdot L(\infty)^t$.

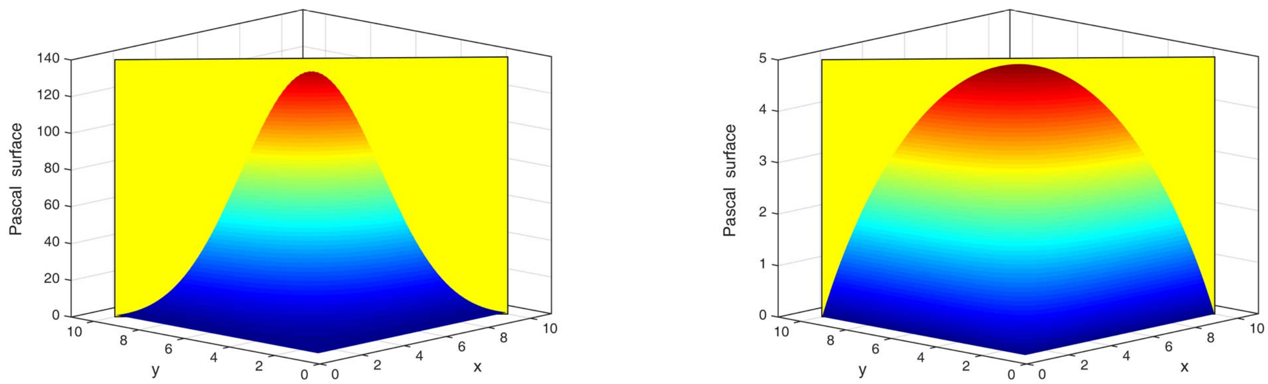


(a) (b)

Figure 3: Discrete Pascal's triangle $P(\infty)$ represented graphically as 3D bars: (a) linear scale; (b) logarithmic scale.

2.2 Pascal's Surface

An extension of the discrete Pascal's triangle to real numbers was introduced by Fowler in [33] ([34] and [35] followed shortly afterwards). This paper took the integer binomial coefficients $\binom{n}{k}$ and extended them to real numbers $\binom{y}{x}$, $0 \leq x \leq y$, by substituting the factorials with equivalent Γ functions (since $\Gamma(x+1) = x!$ for any positive integer x).



(a) (b)

Figure 4: Pascal's surface for $P(\infty)$ presented in Fig.3: (a) linear scale; (b) logarithmic scale.

Fowler mentions in [33] that the resulting surface is Pascal's triangle *"interpolated to a steeply rising ridge."* It is an extension of $L(\infty)_{i,j}$ (presented in Fig. 2), while in this paper we are going to use an extension of $P(\infty)_{i,j}$ (presented in Fig. 3) as:

$$P(x,y) = \binom{x+y}{x} = \frac{(x+y)!}{x!\ y!} = \frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)} \tag{3}$$

$$\ln[P(x,y)] = \ln[\Gamma(x+y+1)] - \ln[\Gamma(x+1)] - \ln[\Gamma(y+1)] \tag{4}$$

which can be seen in Fig. 4, and explore level curves (see [28]) on this surface (for preliminary results see [7]).

3 Approximations

3.1 Gamma Function

Stirling's approximations of factorials are formulas and series first established around 1730 (see [1, 36, 39], and https://en.wikipedia.org/wiki/Stirling%27s_approximation).

Table 1: Approximations of $\Gamma(x+1)$

Author	Formula	Reference(s)
de Moivre	$\mathbf{2.5074} \times \sqrt{x}\left(\frac{x}{e}\right)^x$	1730 [26, 48, 62]
(series)	$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x e^{\left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} \pm \ldots\right)}$	1732 [48, 62]
Stirling	$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x$ [the constant is $\sqrt{2\pi} = \mathbf{2.506628}$]	1730 [74, 78]
(series)	$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} \pm \ldots\right)$	P.-S. Laplace
Burnside	$\sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2}$	1917 [13]
	Stated by de Moivre and Stirling (see [30, 77])	1730 [26, 74]
Ramanujan	$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{\theta}{240x^3}\right)^{\frac{1}{6}}$, $0.3 < \theta < 1$	1920 [66, 67]
Robbins	$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x e^{\frac{1}{12x}}$	1955 [68]
Gosper	$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \sqrt{1 + \frac{1}{6x}}$	1978 [37]
Windschitl	$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left[x \sinh\left(\frac{1}{x}\right)\right]^{\frac{x}{2}}$	2002 [71, 82]
Nemes	$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^2 - \frac{1}{10}}\right)^x$	2010 [59]
Mortici	$\sqrt{2\pi x}\left(\frac{x}{e} + \frac{1}{12ex}\right)^x$	2011 [57]

The first formula is obtained by approximating $\ln(n!)$ with an integral:

$$\ln(n!) = \sum_{k=1}^{n} \ln k \cong \int_1^n \ln x dx = n \ln n - n + 1.$$

Table 2: Approximations of $\ln\left[\Gamma(x+1)\right]$

Approximation	Formula	Year
Integral	$x\ln x - x + 1$	1730
Stirling	$x\ln x - x + \frac{1}{2}\ln(2\pi x)$	1730
Burnside	$\left(x+\frac{1}{2}\right)\ln\left(x+\frac{1}{2}\right) - \left(x+\frac{1}{2}\right) + \frac{1}{2}\ln(2\pi)$	1917 (1730)
Robbins	$\ln x - x + \frac{1}{12x} + \frac{1}{2}\ln(2\pi x)$	1955 (1732)
Windschitl	$\frac{3x}{2}\ln x - x + \frac{x}{2}\ln\left[\sinh\left(\frac{1}{x}\right)\right] + \frac{1}{2}\ln(2\pi x)$	2002

Moving from the prescient formulas of Stirling and de Moivre to those of Ramanujan and beyond, two directions of research can be identified. One was to provide simpler proofs, like, e.g., [53, 58], while the other/major direction was on enhancing accuracy. From this trend we mention [40] (improving on Ramanujan) and [18] (improving on Windschitl), with [83] presenting the latest Windschitl variations, while [56] is more on reviewing/comparisons. All those exhibiting higher accuracies and speeds (of convergence) seem to be trading these for more and more cumbersome formulas (see [16, 45, 46, 65]).

Here we have decided to rely on only a handful of the simpler formulas, and our selection is the one presented in Table 2.

3.2 Binomial coefficients

A particular problem raised by Sir Alexander Cuming in 1721 was to find a good (and fast) approximation of the ratio of the middle term of $(1+1)^{2n}$ to its sum. This was the starting point of de Moivre's quest for an approximation to the central term of the binomial expansion [26]. Other approximations of the binomial coefficients are presented in [47], on the lines of the de Moivre-Laplace approximation, but it is clearly mentioned that using good approximations of the gamma function one could obtain similar accuracies (see [11, 50, 72]).

That is why we shall start with the integral approximation we have presented in Table 2, namely $\ln\left[\Gamma(x+1)\right] \cong x\ln x - x + 1$. Using this particular approximation together with eqs. (3) and (4), we get a first approximation $P_1(x,y)$ of $P(x,y)$:

$$\ln\left[P_1(x,y)\right] = \left[(x+y)\ln(x+y) - (x+y)\right] - (x\ln x - x) - (y\ln y - y) - 1$$

$$= (x+y)\ln(x+y) - (x\ln x + y\ln y) - 1 = \ln\left[\frac{1}{e}\cdot\frac{(x+y)^{(x+y)}}{x^x y^y}\right]$$

$$P_1(x,y) = \frac{1}{e}\cdot\frac{(x+y)^{(x+y)}}{x^x y^y}.$$

Proceeding in a similar fashion for the other four approximations presented in Table 2 (i.e., Stirling, Burnside, Robbins, and Windschitl) we get $P_2(x,y)$, $P_3(x,y)$, $P_4(x,y)$, and $P_5(x,y)$. We omit the details and present these five approximations of $P(x,y)$ in Table 3, as a tabular form allows for easier comparisons.

4 Applications

Table 3: Several approximations for binomial coefficients.

Γ approximation	Binomial approximation
Integral	$P_1(x,y) = \frac{1}{e} \cdot \frac{(x+y)^{(x+y)}}{x^x y^y}$
Stirling	$P_2(x,y) = \frac{1}{\sqrt{2\pi}} \cdot \frac{(x+y)^{(x+y+1/2)}}{x^{(x+1/2)} y^{(y+1/2)}}$
Burnside	$P_3(x,y) = \frac{\sqrt{e}}{\sqrt{2\pi}} \cdot \frac{(x+y+1/2)^{(x+y+1/2)}}{(x+1/2)^{(x+1/2)}(y+1/2)^{(y+1/2)}}$
Robbins	$P_4(x,y) = \frac{1}{\sqrt{2\pi}} \cdot \frac{(x+y)^{(x+y+1/2)}}{x^{(x+1/2)} y^{(y+1/2)}} \cdot e^{-\frac{x^2+xy+y^2}{12xy(x+y)}}$
Windschitl	$P_5(x,y) = \frac{1}{\sqrt{2\pi}} \cdot \frac{(x+y)^{(3x+3y+1)/2}}{x^{(3x+1)/2} y^{(3y+1)/2}} \cdot \frac{\left[\sinh\left(\frac{1}{x+y}\right)\right]^{(x+y)/2}}{\left[\sinh\left(\frac{1}{x}\right)\right]^{x/2}\left[\sinh\left(\frac{1}{y}\right)\right]^{y/2}}$

4.1 Bernstein polynomials

Bernstein polynomials are linear combinations of Bernstein basis polynomials (see also [2]). They were introduced in 1912 [9] as:

$$B_n(f;x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right)\binom{n}{i} x^i (1-x)^{n-i} \tag{5}$$

and used for a constructive proof of the Weierstrass approximation theorem.

4.2 Reliability polynomials

For characterizing network reliability Moore and Shannon introduced reliability polynomials [54], associating them to networks N formed by perfectly reliable *nodes* interconnected by *links* failing with probability $1-p$, hence $Rel(N;p)$ represents the probability of some particular nodes (e.g., source S and terminus T) being connected.

Moore and Shannon expressed this as:

$$Rel(N;p) = \sum_{i=0}^{n} N_i p^i (1-p)^{n-i} \tag{6}$$

where n is the total number of links of N and the coefficients $N_0, N_1, \ldots, N_n$. represent the number of functional sub-networks with i links. Hence this form is in terms of pathsets. Other forms are possible, e.g., in terms of cutsets, or complements of pathsets (see [21]). For more details the interested reader should consult [12, 17, 29, 64].

In this paper we are going to use eq. (6) as matching eq. (5), $N_i = \alpha_i \binom{n}{i}$ being fractions $\alpha_i = f\left(\frac{i}{n}\right) \leq 1$ of the binomial coefficients $\binom{n}{i}$, hence belonging to *Pascal's surface*. For all the simulations we are going to present here we have used the hammock network of $w = l = 3$ $(H_{3,3})$ as an example (for details regarding hammock networks see [22, 24, 25, 44, 54]). If $q = 1-p$, the reliability polynomial for $H_{3,3}$ can be written in Bernstein form as:

$$Rel(H_{3,3};p) = \mathbf{8}p^3q^6 + \mathbf{42}p^4q^5 + \mathbf{84}p^5q^4 + \mathbf{76}p^6q^3 + 36p^7q^2 + 9p^8q + p^9. \tag{7}$$

The level curves of interest are those for $N_i = \mathbf{8}, \mathbf{42}, \mathbf{84}, \mathbf{76}$, the other coefficients being either zeros (the first three), or equal to the binomial ones (the last three).

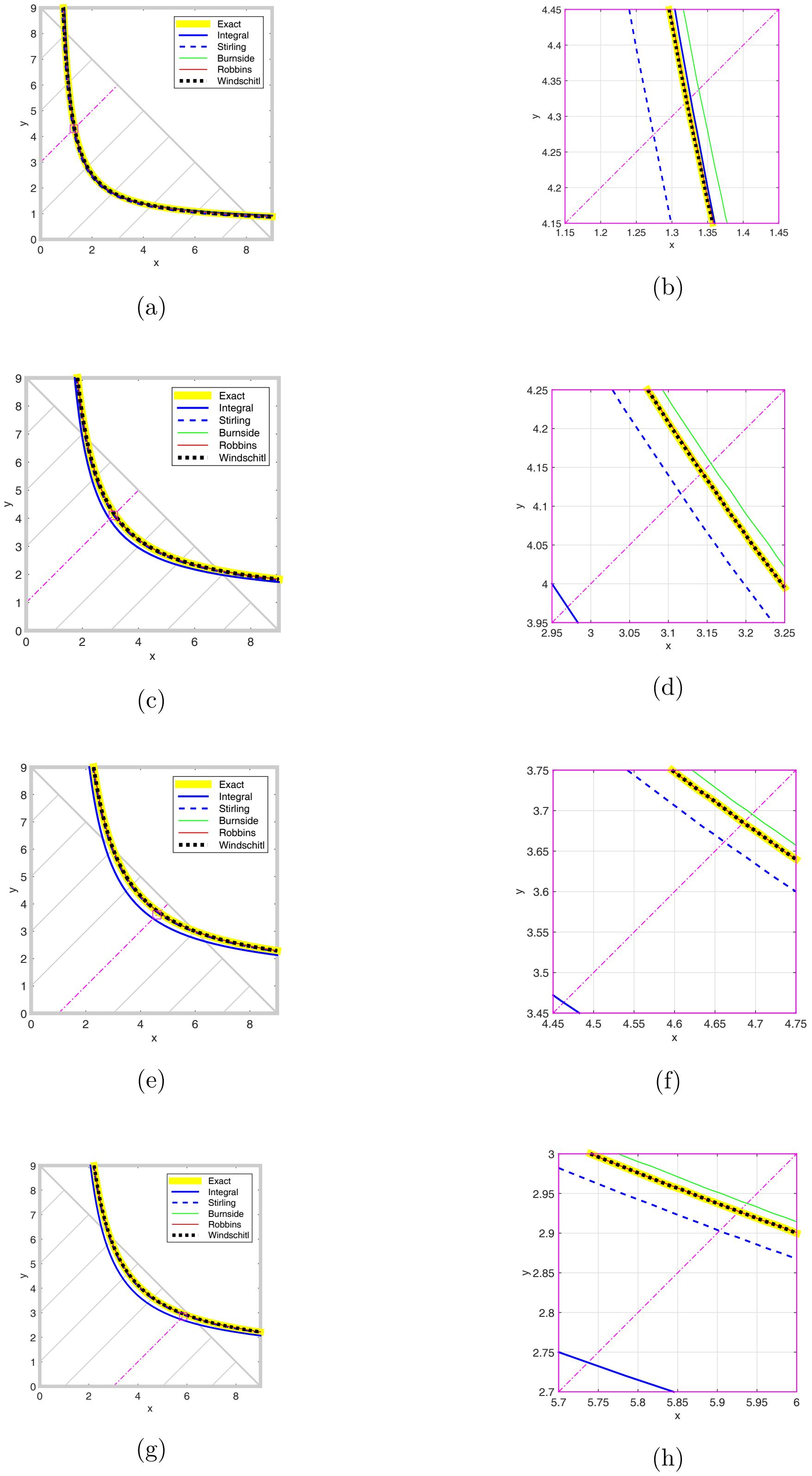


Figure 5: Level curves corresponding to the coefficients of $Rel(H_{3,3}; p)$ using different approximations: (a) **8**; (c) **42**; (e) **84**; (g) **76**; (b), (d), (f), (h) zoom-in (on the corresponding small magenta rectangle from (a), (c), (e), (g)).

We have performed simulations for all the coefficients N_i using the approximations from Table 3, and the results can be seen in Fig. 5. In particular, each binomial coefficient $\binom{9}{i}$ corresponds to a coefficient N_i of the reliability polynomial $Rel(H_{3,3}; p)$. The intersection of Pascal's surface $z = P(x, y)$

(defined by eq. (3)) with the vertical plane of equation $x + y = 9$ is a curve (the blue curve in Fig. 6) containing all the points of coordinates $\left(i,\ 9-i, \binom{9}{i}\right)$. We construct the plane parallel to the plane $x = y$ (the symmetry plane of the surface), through each of these points, for $i = 3, 4, 5, 6$. We denote by $P_i\,(x_i,\ y_i, N_i)$ the point of intersection with the level curve $P(x, y) = N_i$. For completeness, the exact $(x_i,\ y_i, N_i)$ coordinates for $i = 3, 4, \ldots, 9$ are reported in Table 4.

Table 4: The coefficients N_i of $Rel(H_{3,3}; p)$, together with their x_i and y_i coordinates.

i	0 p^0q^9	1 p^1q^8	2 p^2q^7	3 p^3q^6	4 p^4q^5	5 p^5q^4	6 p^6q^3	7 p^7q^2	8 p^8q^1	9 p^9q^0
N_i	0	0	0	**8**	**42**	**84**	**76**	36	9	1
x_i	–	–	–	1.32113	3.14345	4.68527	5.92722	7	8	9
y_i	–	–	–	4.32113	4.14345	3.68527	2.92722	2	1	0

The simulations presented in Fig. 5 reveal that the integral approximation is slightly off, with Stirling and Burnside doing way better, but still leaving room for improvements (see zoom-in in the left column of Fig. 5). Encouragingly, both Robbins and Windschitl approximations appear to be performing fairly well (many more supporting simulations are going to be reported in a follow-up paper).

Following on the 3D geometrical insights/coordinates we have gathered, we decided to compute the Lagrange interpolation polynomial $L\,(x)$ which is linking up the points $(x_i,\ y_i)$, for $i = 3,\ 4,\ \ldots,\ 9$:

$$L\,(x) = 4.4602 - 0.11725x - 0.0096x^2 + 0.02393x^3 - \\ -0.00847x^4 + 0.00083x^5 - 0.00003x^6$$

The smooth curve defined by $\mathbf{r}(x) = (x, L(x), P(x, L(x)))$, $x \in [1, 9]$, belongs to Pascal's surface and is interconnecting the points (x_i, y_i, N_i), $i = 3, 4, \ldots, 9$.

5 Conclusions

In this paper we have considered an extension of Pascal's triangle to a continuous surface, on which the coefficients of any reliability polynomial represented in Bernstein form are to be found at discrete steps (as being on level curves of integer values).

We have approached the level curve problem on this *Pascal's surface* using approximations of the gamma function, and have reviewed the literature on factorial/gamma approximations. We have evaluated through simulations how accurate such approximations would allow us to estimate the level curves. In particular, we have presented for the first time ever simulation results for the reliability polynomial of a small (3×3) hammock network mapped onto *Pascal's surface* (synthesized in Fig. 6). We plan to continue working on this approach, and expect to report many more simulation results in a follow-up paper.

Lastly, we think that Pascal's surface is a mathematical concept which might deserve additional attention (see [60]).

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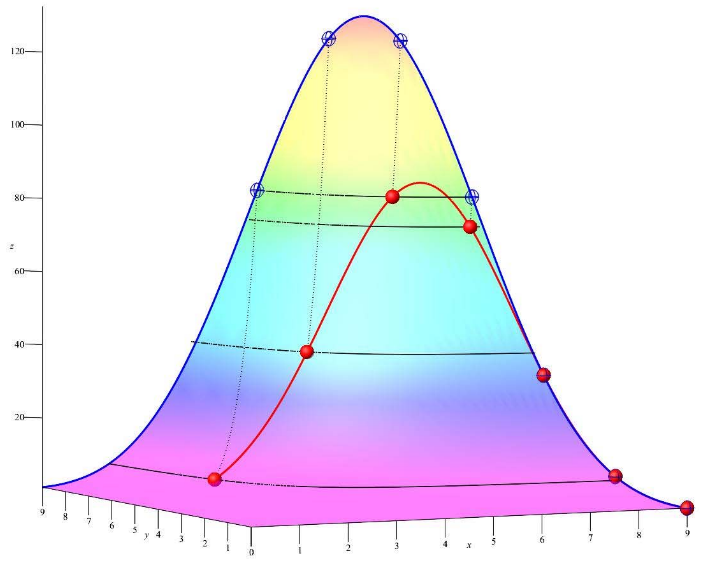

Figure 6: The reliability polynomial $Rel(H_{3,3}; p)$ associated to $H_{3,3}$ represented as a smooth curve (red) on the *Pascal's surface*, connecting the level curves (black) corresponding to the nonzero coefficients (red spheres).

Author contributions

The authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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