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Fixed Point Theory in Fuzzy Normed Linear Spaces: A General View

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Abstract

In this paper we have presented, firstly, an evolution of the concept of fuzzy normed linear spaces, different definitions, approaches as well as generalizations. A special section is dedicated to fuzzy Banach spaces. In the case of fuzzy normed linear spaces, researchers have been working, until now, with a definition of completeness inspired by M. Grabiec's work in the context of fuzzy metric spaces. We propose another definition and we prove that it is much more adequate, inspired by the work of A.George and P. Veeramani. Finally, some important results in fuzzy fixed point theory were highlighted.

Keywords: fuzzy normed linear space, fuzzy Banach spaces, completeness, fixed point theory.

1 Fuzzy normed linear spaces

Nowadays, fixed point theory and functional analysis have an important role not only in mathematics but, most of all, in many and various applications in physics, computer science, economy etc. On the other hand, after L. A. Zadeh had introduced in his famous paper [51] the concept of fuzzy set, many mathematicians considered the fuzzy context as a natural frame within which many notions and mathematical concepts can be generalized.

In 1977, A.K. Katsaras and D.B.Liu [29] introduced the concept of fuzzy topological vector space which led, to what we call today, fuzzy functional analysis. In this new context fixed point theory and operator theory play and will play an important role.

In 1984, A.K. Katsaras [28] introduced, for the first time, the concept of fuzzy normed linear space. The fuzzy norm introduced by A.K. Katsaras is of Minkovsky type associated to an absolutely convex and absorbing fuzzy set. In 1992, C. Felbin [20] introduced a new concept of fuzzy norm, a norm which to each element of a linear space makes it correspond a fuzzy real number. This fuzzy norm induces a fuzzy metric of the Kaleva and Seikalla type [27]. Inspired by S.C. Cheng and J.N. Mordeson [16], in 2003, T. Bag and S.K. Samanta [5] introduced a new definition of fuzzy norm that proves to be a lot more adequate and easier to work with, even if during the following years many mathematicians have tried to simplify it, to polish it, improve it or generalize it. The advantage of this fuzzy norm is that it can be decomposed in a family of semi-norms. Moreover, we have to highlight that this fuzzy norm induces a fuzzy metric of Kramosil and Michálek type [30].

In 2005, R. Saadati and S.M. Vaezpour [43] introduced a new concept of fuzzy norm which induces a fuzzy metric of George and Veeramani type [21]. A generalization of these spaces was done the following year by R. Saadati and J.H. Park [41] who, using continuous t-norms and continuous t-conorms, defined the notion of intuitionistic fuzzy normed spaces.

A comparative study regarding the different concepts of fuzzy norms was realized by T. Bag and S.K. Samanta [9] in 2008. They proved that the fuzzy norm of Bag-Samanta type was similar to that of Katsaras type; it is only that they were defined from two different perspectives. After T. Bag and S.K. Samanta introduced the notion of fuzzy antinorm they showed that to one fuzzy norm of Felbin type it corresponds a pair formed from a fuzzy norm of Bag-Samanta type and a fuzzy antinorm.

In this paper we will use the definition of the fuzzy norm introduced by S. Nădăban and I. Dzitac [37] in 2014.

Definition 1. *Let X be a linear space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$). A fuzzy norm is a mapping $N : X \times [0, \infty) \to [0, 1]$ such that:*

(N1) $N(x, 0) = 0, (\forall)x \in X$;

(N2) $[N(x, \tau) = 1, (\forall)\tau > 0]$ *iff* $x = 0$;

(N3) $N(\lambda x, \tau) = N\left(x, \frac{\tau}{|\lambda|}\right), (\forall)x \in X, (\forall)\tau \geq 0, (\forall)\lambda \in \mathbb{K}^*$;

(N4) $N(x + y, \tau + \varsigma) \geq N(x, \tau) * N(y, \varsigma), (\forall)x, y \in X, (\forall)\tau, \varsigma \geq 0$;

(N5) $(\forall)x \in X, N(x, \cdot)$ *is left continuous and* $\lim_{\tau\to\infty} N(x, \tau) = 1$.

The triple $(X, N, *)$ will be called fuzzy normed linear space (briefly FNLS).

Some remarks are necessary:

1. $*$ is a continuous t-norm (see [48]), i.e. a continuous binary operation

$$* : [0, 1] \times [0, 1] \to [0, 1]$$

   such that $(\forall)s, t, r, u \in [0, 1]$ we have:

   (t1) $s * t = t * s$;

   (t2) $s * 1 = s$;

   (t3) $(s * t) * r = s * (t * r)$;

   (t4) If $s \leq r$ and $t \leq u$, then $s * t \leq r * u$.

   Three important examples of continuous t-norms are $\wedge, \cdot, *_L$ which are defined by: $s \wedge t = \min\{s, t\}$, $s \cdot t = st$ (usual multiplication on $[0, 1]$) and $s *_L t = \max\{s + t - 1, 0\}$ (the Lukasiewicz t-norm).

2. I. Goleţ [23] and C. Alegre and S. Romaguera [2] gave also this definition in the context of real vector spaces.

3. T. Bag and S.K. Samanta [5] gave a similar definition, but they just added two axioms:

   (N6) $N(x, \tau) > 0, (\forall)\tau > 0 \Rightarrow x = 0$;

(N7) $(\forall)x \neq 0, N(x,\cdot)$ is a continuous function and strictly increasing on the subset $\{\tau : 0 < N(x,\tau) < 1\}$ of $\mathbb{R}$.

The results obtained by de T.Bag si S.K. Samanta can be obtained in this more general context as shown in papers [37], [36].

4. This definition is adequate because in this context, as shown in paper [37], X becomes topological vector space. More precisely, for $x \in X, \epsilon \in (0,1)$ and $\tau > 0$ we define the open ball

$$B(X,\epsilon,\tau) = \{y \in X \ : \ N(x-y,\tau) > 1-\epsilon\}.$$

Then

$$\mathcal{T}_N = \{G \subset X \ : \ x \in G \; iff \; (\exists)\tau > 0, (\exists)\epsilon \in (0,1) \; such \; that B(x,\epsilon,\tau) \subseteq G\}$$

is a topology on X.

Moreover, in correlation with this topology, the applications $X \times X \ni (x,y) \to x+y \in X$ and $\mathbb{K} \times X \ni (\lambda,x) \to \lambda x \in X$ are continuous. Therefore X is a topological vector space.

In the particular case in which $* = \wedge$, in the same paper it is shown that X with the topology $\mathcal{T}_N$ becomes a locally convex space.

5. When we work with fuzzy normed linear spaces of type $(X,N,\wedge)$, it is very helpful to us the family of semi-norms $\{||\cdot||_\alpha\}_{\alpha\in(0,1)}$ (see [37]), where

$$||x||_\alpha = \inf\{\tau > 0 \ : \ N(x,\tau) > \alpha\}.$$

2 Generalized fuzzy normed linear spaces

There are several generalizations for FNLS-s. We will present only three that were given less attention in the literature of genre, but that deserve to be studied.

First of all it is about fuzzy ψ-normed spaces, introduced by I. Goleţ [23] in 2010.

Let $\psi : \mathbb{R} \to \mathbb{R}$ such that

1. $\psi(x) = \psi(-x), (\forall)x \in \mathbb{R}$;
2. $\psi(1) = 1$;
3. ψ is strict increasing and continuous on $(0,\infty)$;
4. $\lim\limits_{x\to 0}\psi(x) = 0$; $\lim\limits_{x\to\infty}\psi(x) = \infty$.

There are many examples of such functions: $\psi(x) = |x|$; $\psi(x) = |x|^\alpha, \alpha \in \mathbb{R}_+$ etc.

Let X be a vector space over $\mathbb{R}$. A fuzzy set N in $X \times [0,\infty)$ is called fuzzy ψ-norm if it satisfies $(N1), (N2), (N4), (N5)$ and

(N3)* $N(\lambda x,\tau) = N\left(x, \frac{\tau}{\psi(\lambda)}\right), (\forall)x \in X, (\forall)\tau \geq 0, (\forall)\lambda \in \mathbb{R}$.

The triple $(X,N,*)$ is called fuzzy ψ-normed space.

Another generalization is made by S. Nădăban [35]. Let X be a vector space over $\mathbb{K}$. A fuzzy set P in $X \times \mathbb{R}$ is called fuzzy pseudo-norm if it satisfies:

(PN1) $P(x,\tau) = 0, (\forall)x \in X, (\forall)\tau \leq 0$;

(PN2) $[P(x,\tau) = 1, (\forall)\tau > 0]$ if and only if $x = 0$;

(PN3) $P(\lambda x,\tau) \geq P(x,\tau), (\forall)x \in X, (\forall)\tau \in \mathbb{R}, (\forall)\lambda \in \mathbb{K}, |\lambda| \leq 1$;

(PN4) $P(x+y,\tau+\varsigma) \geq \min\{P(x,\tau), P(y,\varsigma)\}, (\forall)x,y \in X, (\forall)\tau,\varsigma \in \mathbb{R}$;

(PN5) $\lim\limits_{\tau\to\infty} P(x,\tau) = 1, (\forall)x \in X$;

(PN6) If there exists $\alpha_0 \in (0,1)$ such that $P(x,\tau) > \alpha_0, (\forall)\tau \geq 0$, then $x = 0$;

(PN7) $(\forall)x \in X, P(x, \cdot)$ is left continuous on $\mathbb{R}$.

The pair (X, P) will be called fuzzy pseudo-normed linear space.

Finally, in paper [42], the authors worked with C*-AVF normed space. In order to introduce these spaces we consider A be an order commutative C*-algebra and we denote by A^+ the positive cone of A. Let $X \neq \emptyset$. We consider some generalized fuzzy sets in X. These are mappings $\mu : X \to A^+$, which are called C*-algebra valued fuzzy sets, briefly C*-AVF sets. We denote by $0 = \inf A^+$ and $1 = \sup A^+$. We consider generalized t-norms on A^+, namely mappings $\odot : A^+ \times A^+ \to A^+$ such that:

1. $s \odot 1 = s, (\forall)s \in A^+$;
2. $s \odot t = t \odot s, (\forall)s, t \in A^+$;
3. $(s \odot t) \odot r = s \odot (t \odot r), (\forall)s, t, r \in A^+$;
4. If $s_1, s_2, t_1, t_2 \in A^+$ such that $s_1 \leq s_2$ and $t_1 \leq t_2$, then $s_1 \odot t_1 \leq s_2 \odot t_2$.

If in addition, for every sequences $(s_n), (t_n)$ from A^+ converging to $s \in A^+$ and $t \in A^+$, we have that $\lim\limits_{n\to\infty}(s_n \odot t_n) = s \odot t$, we will say that $\odot$ is continuous. These t-norms will be called continuous triangular norms, shortly CTN.

Definition 2. *The triple $(X, N, \odot)$ is called a C*-AVF normed space if X is a vector space over $\mathbb{C}$, $\odot$ is a CTN on A^+ and N is a C*-AVF set on $X \times [0, \infty)$ such that:*

(N1) $N(x, 0) = 0, (\forall)x \in X$;

(N2) $[N(x, \tau) = 1, (\forall)\tau > 0]$ *iff* $x = 0$;

(N3) $N(\lambda x, \tau) = N\left(x, \frac{\tau}{|\lambda|}\right), (\forall)x \in X, (\forall)\tau \geq 0, (\forall)\lambda \in \mathbb{C}^*$;

(N4) $N(x + y, \tau + \varsigma) \geq N(x, \tau) \odot N(y, \varsigma), (\forall)x, y \in X, (\forall)\tau, \varsigma \geq 0$;

(N5) $(\forall)x \in X, N(x, \cdot) : X \to A^+$ *is left continuous and* $\lim\limits_{\tau\to\infty} N(x, \tau) = 1$.

3 Fuzzy Banach spaces

The definition of the convergence of a sequence (x_n) in a FNLS $(X, N, *)$ is natural and it is used by all the researchers. Thus, a sequence (x_n) is said to be convergent if there exists $x \in X$ such that $\lim\limits_{n\to\infty} N(x_n - x, t) = 1, (\forall)t \geq 0$. In this case, x is called the limit of (x_n) and we denote $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$. It is easy to see that $x_N \to x$ iff $x_n \to x$ in the topology $\mathcal{T}_N$.

Regarding the notion of Cauchy sequence, the great majority of the articles use a definition inspired by the work of M. Grabiec in fuzzy metric spaces (see [22]). We will call these sequences G-Cauchy sequences.

Definition 3. *A sequence (x_n) is called G-Cauchy sequence if*

$$\lim_{n\to\infty} N(x_{n+p} - x_n, t) = 1, (\forall)t \geq 0, (\forall)p \in \mathbb{N}^*.$$

Let us note that (x_n) is a G-Cauchy sequence if and only if

$$(\forall)\epsilon \in (0, 1), (\forall)t \geq 0, (\forall)p \in \mathbb{N}^*, (\exists)n_0(\epsilon, t, p) \in \mathbb{N}^* : N(x_{n+p} - x_n, t) > 1 - \epsilon, (\forall)n \geq n_0.$$

We will use, in this paper, a different definition, motivated by the work of A. George and P. Veeramani [21], and we will call these sequences G-V-Cauchy sequences.

Definition 4. *A sequence (x_n) is called G-V-Cauchy sequence if*

$$(\forall)\epsilon \in (0, 1), (\forall)t \geq 0, (\exists)n_0(\epsilon, t) \in \mathbb{N}^* : N(x_{n+p} - x_n, t) > 1 - \epsilon, (\forall)n \geq n_0, (\forall)p \in \mathbb{N}^*.$$

Considering the ideas from [21], we will justify that Definition 3 is the adequate one.

Let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ in $(\mathbb{R}, N, \wedge)$, where $N(x,t) = \frac{t}{t+|x|}$. Thus

$$N(x_{n+p} - x_n, t) = \frac{t}{t + |x_{n+p} - x_n|} = \frac{t}{t + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p}} \to 1,$$

as $n \to \infty$. Hence (x_n) is a G-Cauchy sequence.

As it is natural to consider that $(\mathbb{R}, N, \wedge)$ is complete, we obtain that there exists $x \in \mathbb{R}$ such that $x_n \to x$, i.e. $\lim\limits_{n\to\infty} N(x_n - x, t) = 1, (\forall)t \geq 0$. Therefore $\lim\limits_{n\to\infty} \frac{t}{t+|x_n-x|} = 1, (\forall)t \geq 0$. Hence $|x_n - x| \to 0$, as $n \to \infty$. Thus $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \to x$, which is not true.

Let us show that (x_n) is not a G-V-Cauchy sequence. We suppose that (x_n) is a G-V-Cauchy sequence. Thus, for $\epsilon = \frac{1}{2} \in (0,1), t = \frac{1}{3} > 0, (\exists)n_0 \in \mathbb{N} \ : \ N(x_{n+p} - x_n, t) > 1 - \epsilon, (\forall)n \geq n_0, (\forall)p \in \mathbb{N}^*$. For $p = n$ we obtain that

$$\frac{t}{t + |x_{2n} - x_n|} > 1 - \epsilon \Leftrightarrow \frac{t}{t + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}} > 1 - \epsilon \Leftrightarrow$$

$$\frac{1}{3} > \frac{1}{2}\left(\frac{1}{3} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) \Leftrightarrow \frac{1}{3} > \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

But $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}$ and we have reached a contradiction with the assumption that we made. Therefore (x_n) is not a G-V-Cauchy sequence.

Definition 5. *Let $(X, N, *)$ be a FNLS. $(X, N, *)$ is said to be G-V complete if any G-V-Cauchy sequence in X is convergent to a point of X. A G-V complete FNLS will be called fuzzy Banach space.*

4 Fuzzy continuous mapping and fuzzy bounded linear operators

Let $(X, N_1, *_1)$ and $(Y, N_2, *_2)$ be FNLSs with continuous t-norms $*_1$, $*_2$.

Definition 6. *[6] A mapping $T : X \to Y$ is called fuzzy continuous at $x_0 \in X$, if*

$$(\forall)\varepsilon > 0, (\forall)\alpha \in (0,1), (\exists)\delta = \delta(\varepsilon, \alpha, x_0) > 0, (\exists)\beta = \beta(\varepsilon, \alpha, x_0) \in (0,1) \ \ s.t.$$

$$(\forall)x \in X \ : \ N_1(x - x_0, \delta) > \beta \textit{ we have that } N_2(T(x) - T(x_0), \varepsilon) > \alpha.$$

If T is fuzzy continuous at each point of X, then T is called fuzzy continuous on X.

Definition 7. *[34] A mapping $T : X \to Y$ is said to be fuzzy Lipschitzian on X if there exists $L > 0$ such that*

$$N_2(T(x) - T(y), t) \geq N_1\left(x - y, \frac{t}{L}\right), (\forall)t > 0, (\forall)x, y \in X.$$

If $L < 1$ T is called fuzzy contraction.

We note that a fuzzy Lipschitzian mapping is fuzzy continuous.

Definition 8. *[6] A linear mapping $T : X \to Y$ is called strongly fuzzy bounded on X if there exists $M > 0$ such that*

$$N_2(Tx, t) \geq N_1\left(x, \frac{t}{M}\right), (\forall)t > 0, (\forall)x \in X.$$

In paper [6] the authors also introduce other concepts of boundedness such as weakly fuzzy bounded, uniformly bounded and they define the notion of fuzzy norm of a strongly fuzzy bounded linear operator. T. Bag and S.K. Samanta prove in this context the Hahn-Banach theorem, the open mapping theorem, the closed graph theorem and the uniform boundedness principle.

Fuzzy bounded linear operators have further been studied by many mathematicians (see [34], [45], [12] etc.). Simultaneously fuzzy bounded linear operators in Felbin's type FNLSs have been studied (see [10], [4], [26], [49] etc.).

5 Fuzzy fixed point theory

As it was only natural, the first fixed point theorems in fuzzy context were obtained on fuzzy metric spaces. Thus M. Grabiec [22] introduced the notion of complete fuzzy metric space and extended Banach contraction theorem in fuzzy metric space. His work was continued by J.X. Fang [19] who obtained new fixed point theorems in complete fuzzy metric spaces for some contractive type mappings. After A. George and P. Veeramani [21] modified the definition for a Cauchy sequence, a change that led to another type of completeness for fuzzy metric spaces, many fixed point theorems have been obtained within this new frame. Analyzing the many studies to fixed point theory on fuzzy metric spaces, but, at the same time, as any G-complete fuzzy metric space is G-V-complete, we conclude that the obtaining of fixed point theorems in G-V-complete fuzzy metric spaces is much more valuable. As there are many beautiful and important studies dedicated to the fixed point theorems on fuzzy metric spaces, we will make no further reference. We will refer only to a few recently published papers: [25], [14], [40].

There are significantly less studies regarding the fixed point theorems in FNLSs. Thus, in 2004, J. Xiao and X. Zhu [50] obtained the first fixed point theorems for operators on FNLSs of Felbin type. Then, T. Bag si S.K. Samanta published two articles [7],[8] in 2006 and 2007. In these articles they introduced important notions as fuzzy non-expansive mapping, fuzzy normal structures, l-fuzzy compact sets, sectional fuzzy continuous mappings, but they also proved a Kirk-type fixed point theorem, a Schauder-type theorem and other fixed point theorems were established in FNLSs. Their results were processed and developed by I. Sadeqi and F. Solaty kia [46] with the help of the concept of fuzzy reflexive Banach space.

Other fixed point theorems for self-mappings, coincidence point theorems, coupled fixed point theorems, approximate fixed point or common fixed point theorems for pairs of compatible mappings were obtained in the context of fuzzy normed linear spaces (see [32], [33], [47]) fuzzy n-normed linear spaces (see [18]) or in the frame of intuitionistic fuzzy normed spaces (see [13], [24]) or in generalized fuzzy normed spaces (see [23], [15]).

Fuzzy fixed point theory is a rapidly developing domain. However, most papers in this field do not pay much attention to its applications beyond the theory itself, narrowly understood. So, we feel obliged to refer to the applications as well. The applications of fuzzy fixed point theorems are numerous, both in mathematics as well as in engineering and economics, in problems of approximation, in game theory and many others. Recently, in paper [3] there were presented new fuzzy fixed point theorems in metric spaces and their applications to fuzzy fractals. In paper [39] the authors got a version of the famous Banach contraction principle in fuzzy quasi-metric spaces. Using this result they proved the existence of a fixed point for a contraction on the domain of words. Moreover, this approach led to the existence of solution for some recurrence equations associated to the analysis of Quicksort algorithms and Divide & Conquer algorithms, respectively.Their results were then expanded in the context of intuitionistic fuzzy quasi-metric spaces in paper [44]. In paper [31] the authors proved more fixed point theorems in the context of complete b-metric spaces. As applications, they analyzed the existence of approximate solutions for Fredholm integral inclusions. In paper [1] the authors studied some common fuzzy fixed point in F-metric spaces and applied their result for existence of solution of fuzzy Cauchy problems.

In the following we will present some fixed point theorems in fuzzy Banach spaces. Even if some of them were obtained using axioms (N6) and (N7), and others using completeness in Grabiec's sense, their demonstration in this new context is similar, so we leave it to the reader.

Theorem 9. *[17]* **(Caccioppoli's fixed point theorem).** *Let $(X, N, \wedge)$ be a fuzzy Banach space and $T : X \to X$ satisfying:*

$$(\forall)n \in \mathbb{N}^*, (\exists)k_n > o \ : \ N(T^n(x) - T^n(y), t) \geq N\left(x - y, \frac{t}{k_n}\right), (\forall)x, y \in X, (\forall)t > 0\,.$$

If $k_n \to 0$, then T has a unique fixed point.

Theorem 10. *[17]* **(Kannan's fixed point theorem).** *Let $(X, N, \wedge)$ be a fuzzy Banach space and*

$T : X \to X$. If there exists $k \in (0, 1/2)$ such that

$$N(Tx - Ty, kt) \geq N(x - Tx, t) \wedge N(y - Ty, t), (\forall)x, y \in X, (\forall)t > 0 \ ,$$

then T has a unique fixed point.

Theorem 11. *[47] Let $(X, N, \wedge)$ be a fuzzy Banach space and $f : [0, \infty) \to [0, \infty)$ satisfying:*

1. *f is continuous and nondecreasing;*
2. *$f(t) = 0$ if and only if $t = 0$.*

If $T : X \to X$ satisfying:

$$N(x - y, t) \geq \alpha \Rightarrow N(Tx - Ty, t - f(t)) \geq \alpha, (\forall)x, y \in X, (\forall)t \geq 0, (\forall)\alpha \in (0, 1] \ ,$$

then T has a unique fixed point.

Theorem 12. *[47] Let $(X, N, \wedge)$ be a fuzzy Banach space, $\gamma : (0, \infty) \to [0, 1]$ be a decreasing function and $g : [0, \infty) \to [0, \infty)$ satisfying:*

1. *g is continuous and strictly increasing;*
2. *$g(t) = 0$ if and only if $t = 0$.*

If $T : X \to X$ satisfying:

$$N(x - y, t) \geq \alpha \Rightarrow N(Tx - Ty, g^{-1}(\gamma(t)g(t))) \geq \alpha, (\forall)x, y \in X, (\forall)t \geq 0, (\forall)\alpha \in (0, 1] \ ,$$

then T has a unique fixed point.

Another issue that should be studied is the problem of fuzzy approximate fixed point. This notion was introduced in papers [32], [33], motivated by the work of M. Berinde [11].

Let $(X, N, \wedge)$ be a fuzzy normed linear space and $T : X \to X$. Let $\varepsilon > 0$. A point $u \in X$ is called fuzzy approximate fixed point for T if there exists $\alpha \in (0, 1)$ such that $||u - Tu||_\alpha < \varepsilon$. We will denote by $F^z_\varepsilon(T)$ the set of all fuzzy approximate fixed point for T. We will say that T has the fuzzy approximate fixed point property (f.a.f.p.p.) if

$$(\forall)\varepsilon > 0, F^z_\varepsilon(T) \neq \emptyset.$$

Theorem 13. *[32] Let $(X, N, \wedge)$ be a fuzzy normed linear space, $T : X \to X$ and $\alpha \in (0, 1)$. If*

$$\lim_{n\to\infty} ||T^n x - T^{n+1}x||_\alpha = 0, (\forall)x \in X,$$

then T has the fuzzy approximate fixed point property.

Another approach, with the help of the fuzzy norm would be:

Definition 14. *Let $\varepsilon \in (0, 1)$. A point $u \in X$ is called fuzzy approximate fixed point for T if $N(u - Tu, t) > 1 - \varepsilon, (\forall)t > 0$.*

Theorem 15. *Let $x \in X$. If $\lim_{n\to\infty} N(T^n x - T^{n+1}x, t) = 1, (\forall)t > 0$, then T has the fuzzy approximate fixed point property.*

Proof.

$$\lim_{n\to\infty} N(T^n x - T^{n+1}x, t) = 1, (\forall)t > 0 \Leftrightarrow$$

$$(\forall)\varepsilon \in (0, 1), (\exists)n_0 \in \mathbb{N} \ : \ N(T^n x - T^{n+1}x, t) > 1 - \varepsilon, (\forall)n \geq n_0, (\forall)t > 0.$$

We put $u = T^{n_0}x$. Then $N(u - Tu, t) > 1 - \varepsilon, (\forall)t > 0$. Thus T has the fuzzy approximate fixed point property.

6 Applications in domain theory

Domain theory is the creation of D. Scott who introduced some computational models for computer engineering. Domain theory play an important role in denotational semantics. Let $(X, N, *)$ be a fuzzy normed linear space. We will denote by BX the family of all closed formal balls. More precisely, for $x \in X$, $\alpha \in [0, 1]$ and $t > 0$ we will denote by $[x, \alpha, t]$ the closed ball

$$[x, \alpha, t] = B(x, \alpha, t) = \{y \in X \ : \ N(x - y, t) \geq \alpha\} \ .$$

BX will be $BX := \{[x, \alpha, t] \ : \ x \in X, \alpha \in [0, 1], t > 0\}$.

Let $[x, \alpha, t], [y, \beta, s] \in BX$ such that $0 < \beta < \alpha < 1$ and $t > s$. We define a partial order $\subseteq$ on BX by:

$$[x, \alpha, t] \subseteq [y, \beta, s] \Leftrightarrow N(x - y, t - s) \geq 1 - \alpha + \beta \ .$$

The pair $(BX, \subseteq)$ will be called fuzzy domain normed spaces. In [38] the notion of convergence for a sequence $[x_n, \alpha_n, t_n]$ of elements of BX is introduced. The authors proved that BX is a directed complete partially order set, briefly dcpo, namely every directed subset of BX has supremum if and only if each sequence of BX converge to its maximal element. Finally, the authors proved Banach fixed point theorem on BX:

Theorem 16. *Let $(X, N, \wedge)$ be a fuzzy Banach space and $T : X \to X$ be a fuzzy contraction. Then $g : BX \to BX$, defined by $g[x, \alpha, t] = [Tx, \alpha, Lt]$, has a unique fixed point.*

7 Conclusion and further works

In this paper we have presented an evolution of the concept of fuzzy normed linear space, different approaches and generalizations. Special attention was paid to the concept of completeness. As we have shown, in the vast majority of papers, researchers are working with a definition of completeness inspired by Grabiec's work in the context of fuzzy metric spaces. We argued that a much more appropriate definition would be one that starts from the work of George and Veeramani on fuzzy metric spaces. We have highlighted some important results in the field of fixed point theory in fuzzy normed linear spaces, building fertile ground for study in further papers of some fixed point theorems in fuzzy Banach spaces. Future results will have an impact on Mathematics and numerous applications in Computer Science.

We consider that this paper can be of interest for the researchers who work in the domain of fuzzy functional analysis, in particular in fixed point theory, but at the same time, we hope that the work will be of interest to researchers working in fields such as:

1. integrated solutions in control theory and communications,
2. computational intelligence techniques,
3. decision support systems,

where fuzzy normed linear spaces will be applied in problems such as: fixed point theorems and semantics of programs, in game theory, in problems of approximation, complexity of programs and algorithms, data mining, machine learning.

References

[1] Alansari, M.; Mohammed, S.S. Azam, A. (2020). *Fuzzy Fixed Point Results in F-Metric Spaces with Applications*, Journal of Function Spaces, 5142815.

[2] Alegre, C.; Romaguera, S. (2010). *Characterizations of fuzzy metrizable topological vector spaces and their asymmetric generalization in terms of fuzzy (quasi-)norms*, Fuzzy Sets and Systems, 161(16), 2181–2192.

[3] Andres, J.; Rypka, M. (2019). *On a topological fuzzy fixed point theorem and its application to non-ejective fuzzy fractals II.* Fuzzy Sets and Systems, 370, 79–90.

[4] Bag, T. (2011). *Some fundamental theorems in Felbin's type fuzzy normed linear spaces*, International Journal of Mathematics and Scientific Computing, 1(2), 44–49.

[5] Bag, T.; Samanta, S.K. (2003). *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math., 11(3), 687–705.

[6] Bag, T.; Samanta, S.K. (2005). *Fuzzy bounded linear operators*, Fuzzy Sets and Systems, 151, 513–547.

[7] Bag, T.; Samanta, S.K. (2006). *Fixed point theorems on fuzzy normed linear spaces*, Information Sciences, 176, 2910–2931.

[8] Bag, T.; Samanta, S.K. (2007). *Some fixed point theorems in fuzzy normed linear spaces*, Information Sciences, 177, 3271–3289.

[9] Bag, T.; Samanta, S.K. (2008). *A comparative study of fuzzy norms on a linear space*, Fuzzy Sets and Systems, 159, 670–684.

[10] Bag, T.; Samanta, S.K. (2008). *Fuzzy bounded linear operators in Felbin's type fuzzy normed linear spaces*, Fuzzy Sets and Systems, 159, 685–707.

[11] Berinde, M. (2006). *Approximate fixed point theorems*, Studia Univ. Babes-Bolyai, Mathematica, 51(1), 11–25.

[12] Bînzar, T; Pater, F.; Nădăban, S. (2020). *Fuzzy bounded operators with application to Radon transform*, Chaos, Solitons and Fractals, 141, 110359.

[13] Cancan, M. (2010). *Browder's fixed point theorem and some interesting results in intuitionistic fuzzy normed spaces*, Fixed Point Theory and Applications, 642303.

[14] Chandok, S.; Huang, H.; Radenović, S. (2018). *Some fixed-point results for the generalized F-suzuki type contractions in b-metric spaces*, Sahand Commun. Math. Anal., 11, 81—89.

[15] Chatterjee, S.; Bag, T.; Lee, J.-G. (2020). *Schauder fixed point theorem in generalized fuzzy normed linear spaces*, Mathematics, 8, 1643.

[16] Cheng, S.C.; Mordeson, J.N. (1994). *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Cal. Math. Soc., 86, 429–436.

[17] Das, N.R.; Saha, M.L. (2015). *On fixed points in complete fuzzy normed linear spaces*, Annals of Fuzzy Mathematics and Informatics, 10(4), 515–524.

[18] Elagan, S.K.; Rahmat, M.R.S. (2010). *Some fixed point theorems in fuzzy n-normed spaces*, International J. Math. Combin., 3, 45–56.

[19] Fang, J.X. (1992). *On fixed point theorems in fuzzy metric spaces*, Fuzzy Sets and Systems, 46, 107–113.

[20] Felbin, C. (1992). *Finite dimensional fuzzy normed linear space*, Fuzzy Sets and Systems, 48(2), 239–248.

[21] George, A.; Veeramani, P. (1994). *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, 64(3), 395–399.

[22] Grabiec, M. (1998). *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems, 27, 385–389.

[23] Goleţ, I. (2010). *On generalized fuzzy normed spaces and coincidence point theorems*, Fuzzy Sets and Systems, 161(8), 1138–1144.

[24] Gordji, M.E.; Baghani, H.; Cho, Y.J. (2011). *Coupled fixed point theorems for contractions in intuitionistic fuzzy normed spaces*, Mathematical and Computer Modelling, 54, 1897–1906.

[25] Huang, H.; Carić, B.; Došenović, T.; Rakić, D.; Brdar, M. (2021). *Fixed-Point Theorems in Fuzzy Metric Spaces via Fuzzy F-Contraction*,Mathematics, 9, 641.

[26] Janfada, M.; Baghani, H.; Baghani, O. (2011). *On Felbin's-type fuzzy normed linear spaces and fuzzy bounded operators*,Iranian Journal of Fuzzy Systems, 8(5), 117–130.

[27] Kaleva, O.; Seikkala, S. (1984). *On fuzzy metric spaces*, Fuzzy Sets and Systems, 12(3), 215–229.

[28] Katsaras, A.K. (1984). *Fuzzy topological vector spaces II*, Fuzzy Sets and Systems, 12(2), 143–154.

[29] Katsaras, A.K.; Liu, D.B. (1977). *Fuzzy vector spaces and fuzzy topological vector spaces*, Journal of Mathematical Analysis and Applications, 58, 135–146.

[30] Kramosil, I.; Michálek, J. (1975). *Fuzzy metric and statistical metric spaces*, Kybernetica, 11, 326–334.

[31] Al-Mezel, S.A.; Ahmad, J.; De La Sen, M. (2020). *Some New Fuzzy Fixed Point Results with Applications*, Mathematics, 8, 995.

[32] Mohsenialhosseini, S.A.M.; Mazaheri, H. (2013). *Approximate fixed point theorems in fuzzy norm spaces for an operator*, Advaces in Fuzzy Systems, Article ID 613604.

[33] Mohsenialhosseini, S.A.M.; Mazaheri, H.; Dehghan, M.A. (2013). *Approximate fixed point in fuzzy normed spaces for nonlinear maps*, Iranian Journal of Fuzzy Systems, 10(1), 135–142.

[34] Nădăban, S. (2015). *Fuzzy continuous mapping in fuzzy normed linear spaces*, International Journal of Computers Communications & Control, 10(6), 836–844.

[35] Nădăban, S. (2016). *Fuzzy pseudo-norms and fuzzy F-spaces*, Fuzzy Sets and Systems, 282, 99–114.

[36] Nădăban, S; Bînzar, T; Pater, F. (2017). *Some fixed point theorems for φ-contractive mappings in fuzzy normed linear spaces*, J. Nonlinear Sci. Appl., 10, 5668–5676.

[37] Nădăban, S; Dzitac, I. (2014). *Atomic decompositions of fuzzy normed linear spaces for wavelet applications*, Informatica, 25(4), 643–662.

[38] Rana, A.M.; Buthainah, A.A.A.; Fadhel, F.S. (2015). *On fixed point theorem in fuzzy normed space*, Journal of Al-Nahrain University, 18(4), 138–143.

[39] Romaguera, S.; Sapena, A.; Tirado, P. (2007). *The Banach fixed point theorem in fuzzy quasi-metric spaces with application to the domain of words*, Topology and its Applications, 154(10), 2196–2203.

[40] Secelean, N.A.; Mathew, S.; Wardowski, D. (2019). *New fixed-point results in quasi-metric spaces and applications in fractals theory*, Adv. Differ. Equ., 2019, 177.

[41] Saadati, R; Park, J.H. (2006). *Intuitionistic fuzzy euclidean normed spaces*, Communications in Mathematical Analysis, 1(2), 85–90.

[42] Saadati, R.; Park, C.; O'Regan, D.; Nadaban, S. (2021). *n-Expansively super-homogeneous and (n, k)-contractively sub-homogeneous fuzzy control functions and stability results with numerical examples*, Advances in Difference Equations, 2021:153.

[43] Saadati, R.; Vaezpour, S.M. (2005). *Some results on fuzzy Banach spaces*, Journal of Applied Mathematics and Computing, 17(1-2), 475–484.

[44] Saadati, R.; Vaezpour, S.M.; Cho, Y.J. (2009). *Quicksort algorithm: Application of a fixed point theorem in intuitionistic fuzzy quasi-metric spaces at a domain of words*, Journal of Computational and Applied Mathematics, 228(1), 219–225.

[45] Sadeqi, I.; Solaty kia, F. (2009). *Fuzzy normed linear space and its topological structure*, Chaos, Solitons and Fractals, 40(5), 2576–2589.

[46] Sadeqi, I.; Solaty kia, F. (2009). *Some fixed point theorems in fuzzy reflexive Banach spaces*, Chaos, Solitons and Fractals, 41, 2606–2612.

[47] Saheli, M. (2016). *A contractive mapping on fuzzy normed linear spaces*, Iranian Journal of Numerical Analysis and Optimization, 6(1), 121–136.

[48] Schweizer, A; Sklar, A. (1960). *Statistical metric spaces*, Pacific Journal of Mathematics, 10, 314–334.

[49] Xiao, J.-z.; Zhu, X.-h. (2004). *Some basic theorems for linear operators between fuzzy normed spaces*, Journal of University of Science and Technology of China, 34(4), 414–430.

[50] Xiao, J.-z.; Zhu, X.-h. (2004). *Topological degree theory and fixed point theorems in fuzzy normed linear spaces*, Fuzzy Sets and Systems, 147(3), 437–452.

[51] Zadeh, L. A. (1965). *Fuzzy Sets*, Information and Control, 8, 338–353.

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