INTERNATIONAL JOURNAL OF COMPUTERS COMMUNICATIONS & CONTROL
Special issue on fuzzy logic dedicated to the centenary of the birth of Lotfi A. Zadeh (1921-2017)
Online ISSN 1841-9844, ISSN-L 1841-9836, Volume: 16, Issue: 1, Month: February, Year: 2021
Article Number: 4090, https://doi.org/10.15837/ijccc.2021.1.4090

My Early Researches on Fuzzy Set and Fuzzy Logic

Y. Shi

Yong Shi*
1. School of Electrical and Information Engineering, Southwest Minzu University,
Chengdu, Sichuan, 610041 China
2. Key Lab of Big Data Mining and Knowledge Management Chinese Academy of Sciences,
Beijing, 100190 China
3. Research Center on Fictitious Economy and Data Science,
Chinese Academy of Sciences, Beijing, 100190 China
4. College of Information Science and Technology, University of Nebraska at Omaha,
NE, 68182 USA
*Corresponding author: yshi@unomaha.edu

Abstract

This paper presents the author's works on fuzzy sets and fuzzy systems in early 1980's to celebrate the 100-year birthday of Lotfi A. Zadeh. They were originally published in Chinese. The first part of the paper is about an isomorphic theorem on fuzzy subgroups and fuzzy series of invariant subgroups, which could be a theoretical basis when the multiple-valued computer system will be reconsidered or redeveloped in the future. The second part of the paper describes the convergence theorem of fuzzy integral of type II which was contributed by Wenxiu Zhang and Ruhuai Zhao. Both fuzzy integral of type I developed by M. Sugeno and the fuzzy integral of type II have been playing an important role in the design of various engineering devices for last 40 years.

Keywords: fuzzy subgroup, fuzzy integral, binary numeral system, IQ test, artificial intelligence.

1 Introduction

In 1979, I was 23 years old young man majored in mathematics. One day, I read a newspaper article about fuzzy sets written by Professor Peizhuang Wang. It is the first time I knew the concept of fuzzy sets and my heart was shocked by its fascinating idea of extending {0, 1} to [0, 1]. Under the personal guidance of Professor Wang, I enjoyed much of my spare time as a college sophomore in searching interesting research topics at the "blue ocean" of fuzzy mathematics. I was very crazed about it. To celebrate the 100-year birthday of Lotfi A. Zadeh, the founding father of fuzzy sets and fuzzy logic, I now share these works that I contributed to the literature of fuzzy sets and fuzzy systems in Chinese in early 1980's [9, 10]. The first part of the paper is about an isomorphic theorem on fuzzy subgroups and fuzzy series of invariant subgroups, which could be a theoretical basis when the multiple-valued computer system is revisited and will be developed in the future. The second part of the paper describes the convergence theorem of the fuzzy integral of type II which was contributed by Wenxiu Zhang and Ruhuai Zhao. Both fuzzy integral of type I developed by M. Sugeno and the fuzzy integral of type II have been playing important roles in designing various engineering devices, such as Camcorder and influencing our daily life for last 40 years.

2 Another Isomorphic Theorem on Fuzzy Subgroups And Fuzzy Series of Invariant Subgroups

In 1965, L.A. Zadeh first proposed the concept of fuzzy sets [16], marking the birth of fuzzy mathematics.

The fuzzy group [8] was proposed by L.A. Zadeh in 1971 and attracted the attention of mathematicians at home and abroad. Wu Wangming [14] and Zou Kaiqi [19] have conducted extensive research on fuzzy groups and obtained many important conclusions. Following the homomorphism and several isomorphism theorems of fuzzy groups established in [19], this section establishes another isomorphism theorem and proposes the concept of fuzzy normal subgroup sequence, and discusses the relationship between fuzzy normal subgroup sequence and fuzzy direct product group [19].

First, we give another isomorphism theorem about fuzzy subgroups.

Theorem 1. *Let* X_1 *and* X_2 *be the subgroups of the Abel group* X*,* $\mathrm{X}_{1\,1}, \mathrm{X}_{2\,2}$ *be the normal subgroups of* X_1 *and* X_2 *respectively, and* $\mathrm{G}_{1\,1}, \mathrm{G}_{2\,2}$ *be quasi-normal subgroup of* X_1 *and* X_2*. If*

$$\begin{gathered}\mathrm{X}_{1\,1}=\{\mathrm{x} \mid \mu_{\mathrm{G}_{1\,1}}(x)=\mu_{\mathrm{G}_{1\,1}}(\mathrm{e})=1\} \\ \mathrm{X}_{2\,2}=\{\mathrm{x} \mid \mu_{\mathrm{G}_{2\,2}}(x)=\mu_{\mathrm{G}_{2\,2}}(\mathrm{e})=1\}\end{gathered}$$

then,

$$\begin{gathered}\mathrm{X}_{1\,1}\left(\mathrm{X}_{1\,1} \cap \mathrm{X}_{2\,2}\right) \textit{ is the normal subgroup of } \mathrm{X}_{1\,1}\left(\mathrm{X}_{1\,1} \cap \mathrm{X}_2\right) \\ \mathrm{X}_{2\,2}\left(\mathrm{X}_{2\,2} \cap \mathrm{X}_{1\,1}\right) \textit{ is the normal subgroup of } \mathrm{X}_{2\,2}\left(\mathrm{X}_{2\,2} \cap \mathrm{X}_1\right) \\ \mathrm{X}_{1\,1}\left(\mathrm{X}_{1\,1} \cap \mathrm{X}_2\right) / \mathrm{G}_{1\,1}\left(\mathrm{G}_{1\,1} \cap \mathrm{G}_{2\,2}\right) \simeq \mathrm{X}_{2\,2}\left(\mathrm{X}_{2\,2} \cap \mathrm{X}_1\right) / \mathrm{G}_{2\,2}\left(\mathrm{G}_{2\,2} \cap \mathrm{G}_{1\,1}\right).\end{gathered}$$

is established.

Proof. From the classic group theory [15], we have,

$$\begin{gathered}\mathrm{X}_{1\,1}\left(\mathrm{X}_{1\,1} \cap \mathrm{X}_{2\,2}\right) \text{ be the normal subgroup of } \mathrm{X}_{1\,1}\left(\mathrm{X}_{1\,1} \cap \mathrm{X}_2\right) \\ \mathrm{X}_{2\,2}\left(\mathrm{X}_{1\,1} \cap \mathrm{X}_{2\,2}\right) \text{ be the normal subgroup of } \mathrm{X}_{2\,2}\left(\mathrm{X}_{2\,2} \cap \mathrm{X}_1\right)\end{gathered}$$

From the proposition 2.4 from [19], $\mathrm{G}_{1\,1} \cap \mathrm{G}_{2\,2}$ is the fuzzy quasi-subgroup of $\mathrm{X}_{1\,1} \cap \mathrm{X}_{2\,2}$, thereby it is the fuzzy quasi-subgroup of $\mathrm{X}_{1\,1} \cap \mathrm{X}_2$.

Then,

$$\begin{aligned}\mu_{\mathrm{G}_{1\,1} \cap \mathrm{G}_{2\,2}}\left(\mathrm{xyx}^{-1}\right) & =\min \left\{\mu_{\mathrm{G}_{1\,1}}\left(\mathrm{xyx}^{-1}\right), \mu_{\mathrm{G}_{2\,2}}\left(\mathrm{xyx}^{-1}\right)\right\} \\ & =\min \left\{\mu_{\mathrm{G}_{1\,1}}(\mathrm{y}), \mu_{\mathrm{G}_{2\,2}}(\mathrm{y})\right\} \\ & =\mu_{\mathrm{G}_{1\,1} \cap \mathrm{G}_{2\,2}}(\mathrm{y}) \quad\left(\forall \mathrm{x}, \mathrm{y} \in \mathrm{X}_{1\,1} \cap \mathrm{X}_{2\,2}\right)\end{aligned}$$

$\therefore \mathrm{G}_{1\,1} \cap \mathrm{G}_{2\,2}$ is the fuzzy quasi-normal subgroup of $\mathrm{X}_{1\,1} \cap \mathrm{X}_2$,

Similarly, we can prove that $\mathrm{G}_{2\,2} \cap \mathrm{G}_{1\,1}$ is the fuzzy quasi-normal subgroup of $\mathrm{X}_{2\,2} \cap \mathrm{X}_1$,

It can be derived from Theorem 4 in [19] that $\mathrm{G}_{1\,1}(\mathrm{G}_{1\,1} \cap \mathrm{G}_{2\,2})$ is the fuzzy quasi-normal subgroup of $\mathrm{X}_{1\,1}(\mathrm{X}_{1\,1} \cap \mathrm{X}_2)$

Actually,

$$\mu_{G_{11}(G_{11}\cap G_{22})}(xy^{-1}) = \mu_{[G_{11}(G_{11}\cap G_{22})][G_{11}(G_{11}\cap G_{22})]}(xy^{-1})$$
$$= \operatorname*{Sup}_{z\in X_{11}(X_{11}\cap X_{22})} \min\{\mu_{G_{11}(G_{11}\cap G_{22})}(z), \mu G_{11}(G_{11}\cap G_{22})(z^{-1}xy^{-1})\}$$
$$= \operatorname*{Sup}_{z\in X_{11}(X_{11}\cap X_{22})} \min\{\mu_{G_{11}(G_{11}\cap G_{22})}(z), \operatorname*{Sup}_{u\in X_{11}(X_{11}\cap X_{22})} \min[\mu_{G_{11}}(u), \mu(G_{11}\cap G_{22})(u^{-1}y^{-1})]\}$$
$$= \min\{\mu_{G_{11}(G_{11}\cap G_{22})}(x), \operatorname*{Sup}_{u^{-1}\in X_{11}(X_{11}\cap X_{22})} \min[\mu_{G_{11}}(u^{-1}), \mu_{G_{11}\cap G_{22}}(uy)]\}$$
$$= \min\{\mu_{G_{11}(G_{11}\cap G_{22})}(x), \mu_{G_{11}(G_{11}\cap G_{22})}(y)\}$$

$\therefore G_{11}(G_{11}\cap G_{22})$ is the fuzzy subgroup of $X_{11}(X_{11}\cap X_2)$.
Thus, $\mu_{G_{11}(G_{11}\cap G_{22})}(xyx^{-1}) = \mu_{G_{11}(G_{11}\cap G_{22})}(xx^{-1}y) = \mu_{G_{11}(G_{11}\cap G_{22})}(y)$
$\therefore G_{11}(G_{11}\cap G_{22})$ is the fuzzy normal subgroup of $X_{11}(X_{11}\cap X_2)$.
Then, $\mu_{G_{11}(G_{11}\cap G_{22})}(e) = \operatorname*{Sup}_{y\in X_{11}(X_{11}\cap X_{22})} \min\{\mu_{G_{11}}(y), \mu_{(G_{11}\cap G_{22})}(y^{-1})\}$

$$\geqslant \min\{\mu_{G_{11}}(e), \mu_{G_{11}\cap G_{22}}(e)\} = 1$$

This can prove that $G_{11}(G_{11}\cap G_{22})$ is the fuzzy quasi-normal subgroup of $X_{11}(X_{11}\cap X_2)$.
Similarly, by changing the position of $G_{11}, G_{22}; X_{11}, X_{22}$, we can prove that $G_{22}(G_{22}\cap G_{11})$ is the fuzzy quasi-normal subgroup of $X_{22}(X_{22}\cap X_1)$.

$$\text{Let } H = \left\{x \mid \mu_{G_{11}(G_{11}\cap G_{22})}(x) = \mu_{G_{11}(G_{11}\cap G_{22})}(e) = 1\right\}$$
$$N = \left\{x \mid \mu_{G_{22}(G_{22}\cap G_{11})}(x) = \mu_{G_{22}(G_{22}\cap G_{11})}(e) = 1\right\}$$

Obviously, $H \subseteq X_{11}(X_{11}\cap X_{22})$
Conversely, $\forall x \in X_{11}(X_{11}\cap X_{22})$

$$\mu_{G_{11}(G_{11}\cap G_{22})}(x) = \operatorname*{Sup}_{y\in X_{11}(X_{11}\cap X_{22})} \min\{\mu_{G_{11}}(y), \mu_{G_{11}\cap G_{22}}(y^{-1}x)\}$$
$$= \operatorname*{Sup}_{y\in X_{11}(X_{11}\cap X_{22})} \min\{\mu_{G_{11}}(y), \mu_{G_{11}\cap G_{22}}(xy^{-1})\}$$
$$= \operatorname*{Sup}_{y\in X_{11}(X_{11}\cap X_{22})} \min\{\mu_{G_{11}}(y), \mu_{G_{11}\cap G_{22}}(y^{-1})\}$$
$$= \mu_{G_{11}(G_{11}\cap G_{22})}(e) = 1$$

($\because G_{11}(G_{11}\cap G_{22})$ is proved as the Fuzzy quasi-subgroup)

$$\therefore H \supseteq X_{11}(X_{11}\cap X_{22})$$
$$H = X_{11}(X_{11}\cap X_{22})$$
$$N = X_{22}(X_{22}\cap X_{11})$$

From [19], we have:

$$X_{11}(X_{11}\cap X_2)/X_{11}(X_{11}\cap X_{22}) \simeq X_{11}(X_{11}\cap X_2)/G_{11}(G_{11}\cap G_{22})$$
$$X_{22}(X_{22}\cap X_1)/X_{22}(X_{22}\cap X_{11}) \simeq X_{22}(X_{22}\cap X_1)/G_{22}(G_{22}\cap G_{11})$$

However, in the classical group [15],

$$X_{11}(X_{11}\cap X_2)/X_{11}(X_{11}\cap X_{22}) \simeq X_{22}(X_{22}\cap X_1)/X_{22}(X_{22}\cap X_{11})$$
$$\therefore X_{11}(X_{11}\cap X_2)/G_{11}(G_{11}\cap G_{22}) \simeq X_{22}(X_{22}\cap X_1)/G_{22}(G_{22}\cap G_{11})$$

♯

Let's discuss fuzzy normal subgroup sequence.

Definition 1. *If* $\mathrm{X} = \mathrm{X_o} \supseteq \mathrm{X_1} \supseteq \cdots \supseteq \mathrm{X_K}$ *is the normal subgroup seqence of* X, *and* $\mathrm{G} = \mathrm{G_0}, \mathrm{G_1}, \cdots, \mathrm{G_K}$ *is the fuzzy normal subgroup of* $\mathrm{X} = \mathrm{X_0}, \mathrm{X_1}, \cdots, \mathrm{X_K}$*, then*

$$\mathrm{G} = \mathrm{G_0} \supseteq \mathrm{G_1} \supseteq \cdots \supseteq \mathrm{G_K} \tag{1}$$

is the fuzzy normal subgroup sequence of G *on* X*, and its membership function can be defined as,*

$$\mu_{\mathrm{G}}(\mathrm{x}) = \mu_{\mathrm{G_0}}(\mathrm{x}) \geqslant \mu_{\mathrm{G_1}}(\mathrm{x}) \geqslant \cdots \geqslant \mu_{\mathrm{G}_K}(\mathrm{x})$$

where $\mathrm{X_K} = \{\mathrm{e}\}$ *is the identity group,* $\mathrm{G_K}$ *indicates the fuzzy identify group of* $\mathrm{X_K}$*,* K *is the length of fuzzy normal subgroup sequence.*

If,

$$\mathrm{G} = \mathrm{H_o} \supseteq \mathrm{H_1} \supseteq \cdots \supseteq \mathrm{H_1} \tag{2}$$

is another fuzzy normal subgroup sequence of G *on* X*, if for* $\forall \mathrm{G_1}[\in (1)], \exists \mathrm{H_1}[\in (2)]$*,* $\mu_{\mathrm{G_1}}(\mathrm{x}) = \mu_{\mathrm{H_1}}(\mathrm{x})$ *is established, then* (2) *is the subdivision of* (1)*. Obviously,* $\mathrm{K} \leqslant 1$ *and a fuzzy normal subgroup sequence can be regarded as the subdivision of itself.*

Definition 2. *If a fuzzy normal subgroup sequence of* G *on* X *is no different from its own subdivision, then it is the composite fuzzy subgroup sequence.*

Theorem 2. *If* $\mathrm{G} = \prod_{\mathrm{i}=0}^{\mathrm{n}} \mathrm{A_i}$ *is the direct product of fuzzy groups of* $\mathrm{X} = \prod_{\mathrm{i}=0}^{\mathrm{n}} \mathrm{X_1}$*, and* $\mathrm{A_i}, (\mathrm{i} = 0, 1, \cdots, \mathrm{n})$ *is the fuzzy normal subgroup of* $X_i, (i = 0, 1, \cdots, n)$*, in which* $\mathrm{A_0}$ *is the identify fuzzy group,*

then, G *is the normal fuzzy subgroup of* X*.*

Here, the definition of $\mathrm{X_1}$ *is different from* $\mathrm{X_1}$ *in Definition 1.*

Proof. Let $\mathrm{G_i} = \prod_{\mathrm{j}=0}^{\mathrm{i}} \mathrm{A_j}(\mathrm{i} = 0, 1, \cdots, \mathrm{n})$, $\mathrm{i} < \mathrm{k}$

$$\begin{aligned}
\mu_{\mathrm{G}_i}(\mathrm{x}) &= \min\left\{\mu_{\mathrm{A_0}}(\mathrm{x_0}), \mu_{\mathrm{A_1}}(\mathrm{x_1}), \cdots, \mu_{\mathrm{A_i}}(\mathrm{x_i})\right\} \\
&\geqslant \min\left\{\mu_{\mathrm{A_0}}(\mathrm{x_0}), \mu_{\mathrm{A_1}}(\mathrm{x_1}), \cdots \mu_{\mathrm{A_k}}(\mathrm{x_k})\right\} \\
&= \mu_{\mathrm{G_k}}(\mathrm{x}) \qquad (\mathrm{x_i} \in \mathrm{X_i}(\mathrm{i} = 0, 1, ..., \mathrm{n})) \\
\therefore \quad & \mu_{\mathrm{G_i}}(\mathrm{x}) \geqslant \mu_{\mathrm{G_k}}(\mathrm{x})
\end{aligned}$$

we have

$$\mu_{\mathrm{G_0}}(\mathrm{x}) \geqslant \mu_{\mathrm{G_1}}(\mathrm{x}) \geqslant \cdots \geqslant \mu_{\mathrm{G_k}}(\mathrm{x}) \tag{$*$}$$

$$\textit{or} \qquad \mathrm{G} = \mathrm{G_0} \supseteq \mathrm{G_1} \supseteq \cdots \supseteq \mathrm{G_n}$$

while

$$\begin{aligned}
\mu_{\mathrm{G_i}}(\mathrm{xyx^{-1}}) &= \min\{\mu_{\mathrm{A_0}}(\mathrm{x_0 y_0 x_0^{-1}}), \mu_{\mathrm{A_1}}(\mathrm{x_1 y_1 x_1^{-1}}), \cdots, \mu_{\mathrm{A_i}}(\mathrm{x_i y_i x_i^{-1}})\} \\
&\geqslant \min\{\mu_{\mathrm{A_0}}(\mathrm{Y_0}), \mu_{\mathrm{A_1}}(\mathrm{Y_1}), \cdots, \mu_{\mathrm{A_i}}(\mathrm{Y_i})\} \\
&= \mu_{\mathrm{G_1}}(\mathrm{y}) \qquad (\forall \mathrm{x}, \mathrm{y} \in \prod_{\mathrm{j}=0}^{\mathrm{i}} \mathrm{X_i})
\end{aligned}$$

$\therefore$ $\mathrm{G_i}(\mathrm{i} = 0, 1, \cdots, \mathrm{n})$ is the fuzzy normal subgroup of $\mathrm{X_i} = \prod_{\mathrm{j}=0}^{\mathrm{i}} \mathrm{X_i}(\mathrm{i} = 0, 1, \cdots \mathrm{n})$ respectively.
According to Definition 1, equation $(*)$ is a fuzzy normal subgroup of X. ♯

3 Convergence Theorem of Fuzzy Integral of Type II

Fuzzy integral was setup by M. Sugeno in 1972 [11]. Later, a series of distribution function convergence theorems and Fuzzy integral measure convergence theorems were established by Huang Jin-Li and Zheng Dao-Peng in 1980 [2, 18]. In the same year, the Fuzzy integral set up by M. Sugeno (it is called Fuzzy integral of Type I in this paper) was extended and a new Fuzzy integral (this is named Fuzzy integral Type II in this paper) was introduced by Zhang Wen-xiu and Zhao Ru-huai [17]. Starting with Fuzzy integral of Type I, this section presents the corresponding convergence theorem of Fuzzy integral of type II.

3.1 Preliminary

Assume all fuzzy subsets $\underset{\sim}{F}(\mathrm{X}) = \{\mathrm{A} \mid \mathrm{A} : \mathrm{X} \to [0,1]\}$ of the universe X,

Definition 3. *Fuzzy Monotone $\underset{\sim}{K}$ is the series of subsets on $\underset{\sim}{F}(\mathrm{X})$, if it satisfies:*

- $0 \in \underset{\sim}{K}, 1 \in \underset{\sim}{K}$;
- *If* $\{\mathrm{A}_a\} \subset \underset{\sim}{K}$, $\mathrm{A_a}$ *is monotonic non-decreasing, then* $\lim\limits_{\mathrm{n}\to\infty} \mathrm{A_a} \subset \underset{\sim}{K}$

Definition 4. *The set function $\underset{\sim}{g}(\cdot)$ on the fuzzy monotone $\underset{\sim}{K}$ is the fuzzy measure of Type II if it satisfies:*

- $\underset{\sim}{g}(0) = 0, \underset{\sim}{g}(1) = 1$, *stipulate* $\underset{\sim}{g}(a) = a$ *(a is small enough).*
- *If* $\mathrm{A}_1 \leqslant \mathrm{A}_2, \mathrm{A}_1, \mathrm{A}_2 \in \mathrm{K}$, *then* $\underset{\sim}{g}(\mathrm{A}_1) \leqslant \underset{\sim}{g}(\mathrm{A}_2)$
- *If* $\mathrm{A_a} \uparrow \mathrm{A}, \{\mathrm{A_a}, \mathrm{A}\} \subset \underset{\sim}{K}$, *then* $\lim\limits_{\mathrm{n}\to\infty} \underset{\sim}{g}(\mathrm{A_a}) = \underset{\sim}{g}(\lim\limits_{\mathrm{n}\to\infty} \mathrm{A_a}) = \underset{\sim}{g}(\mathrm{A})$.

Definition 5. *If the monotonous $\underset{\sim}{K}$ on F(x) satisfies:*

- $\mathrm{B} \in \mathrm{K} \Rightarrow \mu_{\mathrm{B}} \in \underset{\sim}{K}$ *(μ_{B} is the characteristic function of classical set B)*
- $\mathrm{A} \in \underset{\sim}{K}, \mathrm{B} \in \mathrm{K} \Rightarrow \mathrm{A} \wedge \mu_{\mathrm{B}} \in \underset{\sim}{K}$

then the two-tuple $(\mathrm{X}, \underset{\sim}{K})$ is fuzzy Measurable Space of Type II. If $\underset{\sim}{g}(\cdot)$ is the fuzzy measure of Type II on $(\mathrm{X}, \underset{\sim}{K})$, then $(\mathrm{X}, \underset{\sim}{K}, \underset{\sim}{g}(\cdot))$ is fuzzy Measure Space of Type II.

Definition 6. *Function $\mathrm{h} : \mathrm{X} \to [0.1]$ is K-Measurable function if for any $a \in [0 \cdot 1]$ have $\mathrm{N_a}(\mathrm{h}) \in \mathrm{K}$.*

Definition 7. *The fuzzy integral of Type II of the K-Measurable function h on the fuzzy set $\mathrm{A} \in \underset{\sim}{K}$ is defined as:*

$$\overset{=}{\int_{\mathrm{A}}} \mathrm{h}(\mathrm{x}) \circ \underset{\sim}{g}(\cdot) = \underset{\mathrm{a}\in[0.1]}{\mathrm{SUP}} \left[\mathrm{a} \wedge \underset{\sim}{g}\left(\mathrm{A} \cap \mu_{\mathrm{N_a}}(\mathrm{h})\right)\right]$$

where ${}_{\mathrm{N_a}}(\mathrm{h}) = \{\mathrm{x} \mid \mathrm{h}(\mathrm{x}) > \mathrm{a}\}$, $\mu_{\mathrm{N_a}}(\mathrm{h})$ *is its characteristic function.*

3.2 Convergence Theorem of Fuzzy Integral of Type II

Theorem 3. *Define $(\mathrm{X}, \underset{\sim}{K}, \underset{\sim}{g}(\cdot))$ is the fuzzy Measure Space of Type II, $\mathrm{A} \in \underset{\sim}{K}$, $\mathrm{h}(\mathrm{x})$ is K-Measurable function, and $\underset{\sim}{g}(\mathrm{A}) \leqslant \sigma(\sigma \geqslant 0)$:*

$$0 \leqslant \overset{=}{\int_{\mathrm{A}}} \mathrm{h}(\mathrm{x}) \circ \underset{\sim}{g}(\cdot) \leqslant \sigma$$

Proof. From 7 in Section 3.1:

$$\overset{=}{\int_A} \mathrm{h}(\mathrm{x}) \circ \underset{\sim}{g}(\cdot) = \underset{\mathrm{a}\in[0,1]}{\mathrm{SUP}} \left[\mathrm{a} \wedge \underset{\sim}{g}\left(\mathrm{A} \cap \mu_{\mathrm{N_a}}(\mathrm{h})\right)\right]$$

$$\begin{aligned} &\because \mathrm{A} \in \mu_{\mathrm{N_a}}(\mathrm{h}) \subset \mathrm{A} \\ &\therefore \underset{\sim}{g}(\mathrm{A} \cap \mu_{\mathrm{N_a}}(\mathrm{h})) \leqslant \underset{\sim}{g}(\mathrm{A}) = \sigma \\ &\forall \mathrm{a} \in [0,1], \mathrm{a} \wedge \underset{\sim}{g}(\mathrm{A} \cap \mu_{\mathrm{N_a}}(\mathrm{h})) \end{aligned}$$

Therefore, $0 \leqslant \overset{=}{\int_A} \mathrm{h}(\mathrm{x}) \circ \underset{\sim}{g}(\cdot) \leqslant \sigma$. ♯

Theorem 4. *(F-Integral Uniform Convergence Theorem of Type I): Define* $(\mathrm{X}, \underset{\sim}{K}, \underset{\sim}{g}(\cdot))$ *is the fuzzy measure space of type I,* $\mathrm{A} \in \underset{\sim}{K}$*,* $\{\mathrm{h_n}(\mathrm{x})\}$ *is a monotonic non-decreasing sequence of K-Measurable function (n=0,1,2,* $\cdots$*).*

$$\textit{If} \quad \lim_{\mathrm{n}\to\infty} \mathrm{h_n}(\mathrm{x}) = \mathrm{h_0}(\mathrm{x}) \textit{converges uniformly on } \mathrm{A}, \text{ then,}$$

$$\lim_{\mathrm{n}\to\infty} \overset{=}{\int_A} \mathrm{h_a}(\mathrm{x}) \circ \underset{\sim}{g}(\cdot) = \overset{=}{\int_A} \mathrm{h_0}(\mathrm{x}) \circ \underset{\sim}{g}(\cdot)$$

Proof. From 7:

$$\overset{=}{\int_A} \mathrm{h_n}(\mathrm{x}) \circ \underset{\sim}{g}(\cdot) = \underset{\mathrm{a}\in[0.1]}{\mathrm{SUP}} \left[\mathrm{a} \wedge \underset{\sim}{g}\left(\mathrm{A} \cap \mu_{\mathrm{N_a}}(\mathrm{h_n})\right)\right]$$

$(\mathrm{n} = 0, 1, 2, \cdots)$

$\because \mathrm{h_n}(\mathrm{x})$is monotonous non-decreasing on A, uniformly converge to $\mathrm{h_0}(\mathrm{x})$

From [17] we have: $\forall\varepsilon > 0,$ it exits a positive number N, such that when $\mathrm{n} > \mathrm{N}, \forall \mathrm{x} \in \mathrm{A}$

$\mu_{\mathrm{N_a}}(\mathrm{h_0}) - \varepsilon < \mu_{\mathrm{N_a}}(\mathrm{h_n}) < \varepsilon + \mu_{\mathrm{N_a}}(\mathrm{h_n})$ is established.

From [18] we have:

$$\mathrm{A} \cap \mu_{\mathrm{N_a}}(\mathrm{h_0}) - \varepsilon < \mathrm{A} \cap \mu_{\mathrm{N_a}}(\mathrm{h_n}) < \varepsilon + \mathrm{A} \cap \mu_{\mathrm{N_a}}(\mathrm{h_0})$$

$$\therefore \underset{\sim}{g}(\mathrm{A} \cap \mu_{\mathrm{N_a}}(\mathrm{h_0})) - \varepsilon < \underset{\sim}{g}(\mathrm{A} \cap \mu_{\mathrm{N_a}}(\mathrm{h_n})) < \varepsilon + \underset{\sim}{g}(\mathrm{A} \cap \mu_{\mathrm{N_a}}(\mathrm{h_0}))$$

$$\therefore \underset{\mathrm{a}\in[0,1]}{\mathrm{SUP}}[\mathrm{a} \wedge \underset{\sim}{g}(\mathrm{A} \cap \mu_{\mathrm{N_a}}(\mathrm{h_0}))] - \varepsilon < \underset{\mathrm{a}\in[0,1]}{\mathrm{SUP}}[\mathrm{a} \wedge \underset{\sim}{g}(\mathrm{A} \cap \mu_{\mathrm{N_a}}(\mathrm{h_n}))] < \varepsilon + \underset{\mathrm{a}\in[0,1]}{\mathrm{SUP}}[\mathrm{a} \wedge \underset{\sim}{g}(\mathrm{A} \cap \mu_{\mathrm{N_a}}(\mathrm{h_0}))]$$

That is, $\overset{=}{\int_A} \mathrm{h_0}(\mathrm{x})\mathrm{o}\, \underset{\sim}{g}(\cdot) - \varepsilon < \overset{=}{\int_A} \mathrm{h_n}(\mathrm{x})\mathrm{o}\, \underset{\sim}{g}(\cdot) < \varepsilon + \overset{=}{\int_A} \mathrm{h_0}(\mathrm{x})\mathrm{o}\, \underset{\sim}{g}(\cdot)$

$\therefore \forall\varepsilon \geq 0,$ exit $\mathrm{N} = \mathrm{N}(\varepsilon),$ when $\mathrm{n} \geq \mathrm{N}, \forall \mathrm{x} \in \mathrm{A}$

$$|\overset{=}{\int_A} \mathrm{h_n}(\mathrm{x})\mathrm{o}\, \underset{\sim}{g}(\cdot) - \overset{=}{\int_A} \mathrm{h_0}(\mathrm{x})\mathrm{o}\, \underset{\sim}{g}(\cdot)| < \varepsilon$$

Condition (*): Assume $(\mathrm{X}, \underset{\sim}{\mathrm{K}}, \underset{\sim}{\mathrm{g}}(\cdot))$ is the fuzzy measurable space of type II, $\underset{\sim}{\mathrm{K}}$ have complementary elements (that is close to set subtraction). $\mathrm{A} \in \underset{\sim}{K}, \mathrm{h_n}(\mathrm{x})(\mathrm{n} = 0, 1, 2, \cdots)$ is K-measurable function, and $\underset{\sim}{\mathrm{g}}(\cdot)$ is A-addrable. (That is $\forall \mathrm{A_1}, \mathrm{A_2}, \mathrm{A_1} \cup \mathrm{A_2} \in \underset{\sim}{\mathrm{K}}, \underset{\sim}{\mathrm{g}}(\mathrm{A_1} \cup \mathrm{A_2}) = \underset{\sim}{\mathrm{g}}(\mathrm{A_1}) \vee \underset{\sim}{\mathrm{g}}(\mathrm{A_2})$)

By introducing the condition (*):

$3 \cdot 1$. The additivity of $\underset{\sim}{\mathrm{K}}$, that is $\forall \mathrm{A_n} \in \underset{\sim}{\mathrm{K}}, \mathrm{n} \in \mathrm{N}^+$, we have:

$$\underset{\sim}{g}(\bigvee_{\mathrm{n}=1}^{\infty} \mathrm{A_n}) = \bigvee_{\mathrm{n}=1}^{\infty} \underset{\sim}{g}(\mathrm{A_n})$$

$3\cdot 2$. $\forall \mathrm{n} \in \mathrm{N}^{+}, \mathrm{h_n}-\mathrm{h_0}, \mathrm{h_0}-\mathrm{h_n}$ are K-measurable functions, and $|\mathrm{h_n}(\mathrm{x})-\mathrm{h_0}(\mathrm{x})|$ is also a K-measurable function.

$3\cdot 3$. Definition: If $\forall\varepsilon>0$, we have: $\lim\limits_{n\to\infty} \underset{\sim}{g}\left(\{\mathrm{x}||\mathrm{h_n}(\mathrm{x})-\mathrm{h_0}(\mathrm{x})||\geq\varepsilon\}\wedge \mathrm{A}\right\rangle=0$, then $\{\mathrm{h_n}(\mathrm{x})\}$ converges to $\mathrm{h_0}(\mathrm{x})$ according to fuzzy measure of type I.

$3\cdot 4$. Definition: If $\sigma\geq 0$, existing $\mathrm{A}_\sigma \in \underset{\sim}{K}, \mathrm{A}_\sigma\subseteq \mathrm{A}, \underset{\sim}{g}(\mathrm{A}_\sigma)<\sigma$ makes $\lim\limits_{n\to\infty}\mathrm{h_a}(\mathrm{x})=\mathrm{h_0}(\mathrm{x})$ (uniformly converge on $\mathrm{A}-\mathrm{A}_\sigma$), then the fuzzy of $\{\mathrm{h_n}(\mathrm{x})\}$ is nearly uniformly convergent to $\mathrm{h_0}(\mathrm{x})$ on A.

$3\cdot 1$ and $3\cdot 2$ are proved as follows.

i>. From Definition 4:

$$\underset{\sim}{g}\left(\overset{\infty}{\underset{n=1}{\vee}}\mathrm{A_a}\right)=\underset{\sim}{g}\left(\lim_{n\to\infty}\mathrm{A_a}\right)=$$
$$=\lim\nolimits_{n\to\infty}\underset{\sim}{g}\left(\mathrm{A_n}\right)=\overset{\infty}{\underset{n=1}{\vee}}\underset{\sim}{g}\left(\mathrm{A_a}\right)$$

$3\cdot 1$ is proved.

ii>. From [17], set $\mathrm{h_{0n}}=\sum\limits_{\mathrm{i}=1}^{\mathrm{n}}\mathrm{a_i}\mu_{\mathrm{B_i}}, \mathrm{h_{kn}}=\sum\limits_{\mathrm{j}=1}^{\mathrm{n}}\mathrm{b_j}\mu_{\mathrm{c_j}}$, where $\mathrm{a_i},\mathrm{b_j}\in[0,1], \mathrm{B_i}, \mathrm{C_j}\in \mathrm{K}(\mathrm{i},\mathrm{j}=1,\cdots,\mathrm{n})$

$\because \mathrm{h_{0n}}-\mathrm{h_{kn}}=\sum\limits_{\mathrm{i\cdot j}}(\mathrm{a_i}-\mathrm{b_j})\mu_{\mathrm{Bi}}\cap \mathrm{c_j}$ are still simple functions.

$\therefore \lim\limits_{n\to\infty}[\mathrm{h_{0n}}-\mathrm{h_{kn}}]=\mathrm{h_0}-\mathrm{h_k}$, so $\mathrm{h_0}-\mathrm{h_k}$ is K-measurable functions. $\mathrm{h_k}-\mathrm{h_0}, |\mathrm{h_k}-\mathrm{h_0}|$ can be proved similarly. $3\cdot 2$ is proved.

♯

Theorem 5. *Suppose that under the condition (*), the necessary and sufficient condition of* $\{\mathrm{h_n}(\mathrm{x})\}$ *to converge to* $\mathrm{h_0}(\mathrm{x})$ *according to fuzzy measure of type II is that* $\{\mathrm{h_n}(\mathrm{x})\}$ *F-nearly uniformly converges to* $\mathrm{h_0}(\mathrm{x})$ *on* A.

Proof. For sufficiency, supposing that $\lim\limits_{n\to\infty}\mathrm{h_n}(\mathrm{x})=\mathrm{h_0}(\mathrm{x})$, (which converges uniformly on $\mathrm{A}-\mathrm{A}_\sigma$), that is $\forall\sigma>0$, when $\mathrm{A}_\sigma\in\underset{\sim}{K}, \underset{\sim}{g}(\mathrm{A_a})\leqslant\sigma, \mathrm{A}_\sigma\subset\mathrm{A}$ exists, $\forall\varepsilon\geq 0$, exists $\mathrm{N}(\varepsilon,\sigma)\in\mathrm{N}^{+}$, when $\mathrm{n}>\mathrm{N}(\varepsilon,\sigma), \forall \mathrm{x}\in\mathrm{A}-\mathrm{A}_\sigma, |\mathrm{h_n}(\mathrm{x})-\mathrm{h_0}(\mathrm{x})|<\varepsilon$. We have,

$$\begin{aligned}
&\underset{\sim}{g}\left(\{\mathrm{x}||\mathrm{h_a}(\mathrm{x})-\mathrm{h_0}(\mathrm{x})|\geqslant\varepsilon\}\wedge(\mathrm{A}-\mathrm{A}_\sigma)\right)=0\\
&\therefore\forall\varepsilon,\sigma>0, \text{exists } \mathrm{N}(\varepsilon,\sigma)\in\mathrm{N}^{+}, \mathrm{n}>\mathrm{N}(\varepsilon,\sigma),\\
&\quad\underset{\sim}{g}(\{x||\mathrm{h_a}(\mathrm{x})-\mathrm{h_0}(\mathrm{x})|\geqslant\varepsilon\}\wedge\mathrm{A})\\
&\quad=\underset{\sim}{g}(\{x||\mathrm{h_a}(\mathrm{x})-\mathrm{h_0}(\mathrm{x})|\geqslant\varepsilon\}\wedge\mathrm{A}_\sigma)\\
&\quad\vee\underset{\sim}{g}(\{x||\mathrm{h_a}(\mathrm{x})-\mathrm{h_0}(\mathrm{x})|\geqslant\varepsilon\}\wedge(\mathrm{A}-\mathrm{A}_\sigma))\leqslant\sigma\vee 0=\sigma
\end{aligned}$$

Sufficiency is proved.

For necessity, supposing $\{\mathrm{h_n}(\mathrm{x})\}$ converges to $\mathrm{h_0}(\mathrm{x})$ according to fuzzy measure of type II on A, that is $\forall\varepsilon=\frac{1}{\mathrm{P}}, \mathrm{P}\in\mathrm{N}^{+}, \forall\sigma>0$, there exists $\mathrm{N}(\mathrm{p},\sigma)\in\mathrm{N}^{+}$, such that when $\mathrm{n}>\mathrm{N}(\mathrm{p},\sigma)$,

$$\underset{\sim}{g}\left(\{\mathrm{x}||\mathrm{h_a}(\mathrm{x})-\mathrm{h_o}(\mathrm{x})\ |\geqslant\varepsilon\}\wedge\mathrm{A}\right)\leqslant\sigma \tag{3}$$

$$\text{Set:} \quad A_{n,p} \triangleq \{x||h_n - h_0| \geqslant \frac{1}{P}\} \wedge A$$

$$A_\sigma \triangleq \bigvee_{p=1}^{\infty} \bigvee_{n=N(p,\sigma)}^{\infty} A_{n,p}$$

$$\text{Apparently,} \quad A_\sigma \subseteq A, A_\sigma \in \underset{\sim}{K}$$

$From(3)$:

$$\underset{\sim}{g}(A_\sigma) = \bigvee_{p=1}^{\infty} \bigvee_{n=N(p,\sigma)}^{\infty} \underset{\sim}{g}(A_{n,p}) \leqslant \sigma$$

Then,

$$A - A_\sigma = A \wedge [\bigvee_{p=1}^{\infty} \bigvee_{n=N(p,\sigma)}^{\infty} \{x||h_n - h_0| < \frac{1}{P}\}]$$

$$\therefore \forall \varepsilon > 0, \text{set} \frac{1}{P} \leqslant \varepsilon, P \in N^+, \text{from } (3), \forall \sigma > 0,$$

$$\text{exists } N(p,\sigma) \in N^+, \text{when } n > N(P,\sigma),$$

$$\forall x \in A - A_\sigma \quad \text{have} : |h_n(x) - h_0(x)| < \frac{1}{P} \leqslant \varepsilon$$

Necessity is proved. ♯

Theorem 6. *(The Convergence Theorem of Fuzzy Integral of Type II according to Fuzzy Measure of Type I): Assuming under condition (*),* $\{h_n(x)\}$ *F-nearly uniformly converges to* $h_0(x)$ *on* A*, then:*

$$\lim_{n\to\infty} \overset{=}{\int_A} h_n(x) \circ \underset{\sim}{g}(\cdot) = \overset{=}{\int_A} h_0(x) \circ \underset{\sim}{g}(\cdot)$$

Proof. For $\{h_n(x)\}$ F-nearly uniformly converges to $h_0(x)$ on A, that is,
$\forall \sigma > 0$, exists $A_\sigma \in \underset{\sim}{K}, A_\sigma \subseteq A, \underset{\sim}{g}(A_\sigma) \leqslant \sigma$,
$\{h_n(x)\}$ uniformly converges to $h_0(x)$ on $A - A_\sigma$, form 4:

$$\lim_{n\to\infty} \overset{=}{\int_{A-A_\sigma}} h_n(x) \circ \underset{\sim}{g}(\cdot) = \overset{=}{\int_{A-A_\sigma}} h_0(x) \circ \underset{\sim}{g}(\cdot)$$

That is, $\forall \sigma > 0$, exists $A_\sigma \in \underset{\sim}{K}, A_\sigma \subseteq A, \underset{\sim}{g}(A_\sigma) \leqslant \sigma$, exists $N(\sigma) \in N^+$
when $n > N(\sigma)$,

$$\left| \overset{=}{\int_{A-A_\sigma}} h_n(x) \circ \underset{\sim}{g}(\cdot) - \overset{=}{\int_{A-A_\sigma}} h_o(x) \circ \underset{\sim}{g}(\cdot) \right| < \sigma \tag{4}$$

Then, according to 3 and $\underset{\sim}{g}(A_\sigma) \leqslant \sigma$, it can be derived that:

$$0 \leqslant \overset{=}{\int_{A-A_\sigma}} h_n(x) \underset{\sim}{g}(\cdot) \leqslant \sigma \quad (n = 0, 1, 2, \cdots) \tag{5}$$

From (3) and (5),

$$\left| \overset{=}{\int_{A}} h_n(x) \circ \underset{\sim}{g}(\cdot) - \overset{=}{\int_{A-A_\sigma}} h_o(x) \circ \underset{\sim}{g}(\cdot) \right| \leqslant \sigma \quad (n = 0, 1, 2, \cdots,) \tag{6}$$

Actually:

$$\begin{aligned}
&\because \quad 0 \leqslant \overset{=}{\int\limits_{A-A_\sigma}} h_n(x)\underset{\sim}{g}(\cdot) \leqslant \sigma \\
&\therefore \quad \underset{\sim}{g}\left(A \cap \mu_{N_a}\left(h_n\right)\right)-g\left(A_\sigma \cap \mu_{N_a}\left(h_n\right)\right)-\sigma \\
&\qquad \leqslant \underset{\sim}{g}\left(A \cap \mu_{N_a}\left(h_n\right)\right)-g\left(A_\sigma \cap \mu_{N_a}\left(h_n\right)\right) \\
&\qquad \leqslant \underset{\sim}{g}\left(A \cap \mu_{N_a}\left(h_n\right)\right) \\
&\qquad \underset{\sim}{g}\left(A \cap \mu_{N_a}\left(h_n\right)\right)-g\left(A_\sigma \cap \mu_{N_a}\left(h_n\right)\right)+\sigma \\
&\qquad \therefore \underset{\sim}{g}\left(\left(A-A_\sigma\right) \cap \mu_{N_a}\left(h_n\right)\right)-\sigma \leqslant \underset{\sim}{g}\left(A \cap \mu_{N_a}\left(h_n\right)\right) \\
&\qquad \leqslant \underset{\sim}{g}\left(\left(A-A_\sigma\right) \cap \mu_{N_a}\left(h_n\right)\right)+\sigma \\
&\qquad \forall a \in[0,1], a \wedge \underset{\sim}{g}\left(\left(A-A_\sigma\right) \cap \mu_{N_a}\left(h_n\right)\right)-\sigma \\
&\qquad \leqslant \underset{\sim}{g}\left(A \cap \mu_{N_a}\left(h_n\right)\right) \\
&\qquad \leqslant \underset{\sim}{g}\left(\left(A-A_\sigma\right) \cap \mu_{N_a}\left(h_n\right)\right)+\sigma
\end{aligned}$$

$$\begin{aligned}
&\text{Thus, } \overset{=}{\int\limits_{A-A_\sigma}} h_n(x) \circ g(\cdot)-\sigma=\underset{a \in(0,1)}{\mathrm{SUP}}\left[a \wedge g\left(\left(A-A_\sigma\right) \cap \mu_{N_a}\left(h_n\right)\right)\right]-\sigma \\
&\leqslant \overset{=}{\int\limits_{A-A_\sigma}} h_n(x) \circ g(\cdot)-\sigma \leqslant \underset{a \in(0,1)}{\mathrm{SUP}}\left[a \wedge g\left(\left(A-A_\sigma\right) \cap \mu_{N_a}\left(h_n\right)\right)\right]+\sigma \\
&=\overset{=}{\int\limits_{A-A_\sigma}} h_n(x) \circ g(\cdot)+\sigma \\
&\text{That is, } \left|\overset{=}{\int\limits_{A}} h_n(x) \circ g(\cdot)-\overset{=}{\int\limits_{A-A_\sigma}} h_n(x) \circ g(\cdot)\right| \leqslant \sigma \quad(n=0,1,2, \cdots)
\end{aligned}$$

Combine (4) and (6):

$$\begin{aligned}
&\left|\overset{=}{\int\limits_{A}} h_n(x) \circ \underset{\sim}{g}(\cdot)-\overset{=}{\int\limits_{A}} h_o(x) \circ \underset{\sim}{g}(\cdot)\right| \leqslant\left|\overset{=}{\int\limits_{A}} h_n(x) \circ \underset{\sim}{g}(\cdot)-\overset{=}{\int\limits_{A-A_\sigma}} h_n(x) \circ \underset{\sim}{g}(\cdot)\right| \\
&+\left|\overset{=}{\int\limits_{A-A_\sigma}} h_n(x) \circ \underset{\sim}{g}(\cdot)-\overset{=}{\int\limits_{A-A_\sigma}} h_0(x) \circ \underset{\sim}{g}(\cdot)\right| \\
&+\left|\overset{=}{\int\limits_{A-A_\sigma}} h_0(x) \circ \underset{\sim}{g}(\cdot)-\overset{=}{\int\limits_{A}} h_0(x) \circ \underset{\sim}{g}(\cdot)\right| \\
&\leqslant\left|\overset{=}{\int\limits_{A-A_\sigma}} h_n(x) \circ \underset{\sim}{g}(\cdot)-\overset{=}{\int\limits_{A-A_\sigma}} h_0(x) \circ \underset{\sim}{g}(\cdot)\right|+2 \sigma \leqslant 3 \sigma \\
&\text{Therefore: } \lim_{n \rightarrow \infty} \overset{=}{\int\limits_{A}} h_n(x) \circ \underset{\sim}{g}(\cdot)=\overset{=}{\int\limits_{A}} h_0(x) \circ \underset{\sim}{g}(\cdot)
\end{aligned}$$

♯

Corollary 7. *Assuming under the condition (*),* $\{\mathrm{h_n(x)}\}$ *converges to* $\mathrm{h_0(x)}$ *according to fuzzy measure of type II on A, then* $\lim\limits_{\mathrm{n}\to\infty} \overset{=}{\int\limits_{\mathrm{A}}} \mathrm{h_n(x)} \circ \underset{\sim}{g}(\cdot) = \overset{=}{\int\limits_{\mathrm{A}}} \mathrm{h_0(x)} \circ \underset{\sim}{g}(\cdot)$

Proof. It can be proved according to 5 ♯

From the above discussion, we can obtain:

Theorem 8. *Define (X, B, P) as the probability space,* $\mathrm{h : X \to [0, 1]}$ *is K-measurable function,* $\mathrm{F \subset K}, \underset{\sim}{P}(\cdot)$ *represents fuzzy measure of type II on* $\underset{\sim}{K}$*, then the inequality holds:*

$$\left| \overset{=}{\int\limits_{\mu_{\mathrm{F}}}} \mathrm{h(x)} \circ \underset{\sim}{P}(\cdot) - \int\limits_{\mathrm{F}} \mathrm{h(x)dp} \right| \leqslant \frac{1}{4}$$

Proof. From [17], we know:

$$\overset{=}{\int\limits_{\mu_{\mathrm{F}}}} \mathrm{h(x)} \circ \underset{\sim}{P}(\cdot) \Longleftrightarrow \overset{-}{\int\limits_{\mathrm{F}}} \mathrm{h(x)P(\cdot)}$$

Then, according to [12] or [13]:

$$\left| \overset{=}{\int\limits_{\mu_{\mathrm{F}}}} \mathrm{h(x)} \circ \underset{\sim}{P}(\cdot) - \int\limits_{\mathrm{F}} \mathrm{h(x)dp} \right| = \left| \overset{-}{\int\limits_{\mathrm{F}}} \mathrm{h(x)} \circ \mathrm{P}(\cdot) - \int\limits_{\mathrm{F}} \mathrm{h(x)dp} \right| \leqslant \tfrac{1}{4}$$

♯

4 Conclusion

The birth of fuzzy sets and fuzzy logic can be regarded as a landmark event of mathematics and computing technology in human history.

In 1703, Gottfried Leibniz published his paper Explication de l'Arithmétique Binaire [3], which is translated into English as the "Explanation of the binary arithmetic". He invented 0,1-binary numeral system and explained its connection with the ancient Chinese figures of Fu Xi. As the simplified version of decimal numeral system, Leibniz's binary system gradually became the basis of the current computer design. It changed our human life dramatically for the last 300 years. The recent achievement of Google's AlphaGo and AlphaGo Zero have demonstrated that the binary numeral system-based computer can easily outperform human beings by massive calculation in a short time. However, if a computer like AlphaGo or even a super-computer plays with three persons in Chinese Mahjong, when one human player sends an eye contact to another human player, the machine cannot figure out how to calculate the human signal. This could partially cause by the simple binary numeral system that has difficulty to figure out the human contact. In 2014-2017, I and my student Feng Liu with other colleagues conducted an interesting research by using Human IQ test to measure machine. According to our finding, the IQ test for virtual assistants shows that even the best one, such as Google, still is not smarter than a 6-year-old human [1, 4, 5, 6, 7]. This means that there is a long way to go for the machine's intelligence to catch up that of our human beings. Perhaps, someday in the future when we use fuzzy logic (multiple numeral system) to design a new computer, its computing power of handling complex calculation can easily catch up and solve the human contact problem. By that time, artificial intelligence will be smarter to understand human being. From my point of view, I strongly believe that fuzzy sets and fuzzy logic invented by Lotfi A. Zadeh will deeply influence our science, mathematics and society in the future which our generation of human beings cannot imagine today. Happy 100-year birthday to Lotfi A. Zadeh! He is living at our mind forever!

Acknowledgements

The author deeply appreciates Prof. Peizhuang Wang's mentorship for the last 40 years. He also thanks Mr.Pei Quan for his great assistance in preparing this paper. This work was partially supported by the National Natural Science Foundation of China (No.71932008).

References

[1] Hands, J. (2019). Cosmosapiens: Human evolution from the origin of the universe. *Abrams*, 2019.

[2] Huang, J.; Zheng, D. (1980). The convergence theorem of fuzzy integral of distribution function *Modern Cybernetics Information*, 1980. (In Chinese)

[3] Leibniz, G. (1879). Explication de I'Arithm6tique Binaire. *Die Mathematische Schriften, ed*, 1879.

[4] Liu, F.; Shi, Y. (2014). The search engine IQ test based on the Internet IQ evaluation algorithm. *Procedia Computer Science*, 31, 1066-1073, 2014.

[5] Liu, F.; Shi, Y.; Liu, Y. (2017). Intelligence quotient and intelligence grade of artificial intelligence. *Annals of Data Science*, 4(2), 179–191, 2017.

[6] Liu, F.; Shi, Y.; Wang, B. (2015) World search engine IQ test based on the internet IQ evaluation algorithms. *International Journal of Information Technology & Decision Making*, 14(02), 221-237, 2015.

[7] *MIT Technology Review*. October 13, 2017.

[8] Rosenfeld, A. (1971). Fuzzy groups. *Journal of mathematical analysis and applications*, 35(3), 512-517, 1971.

[9] Shi, Y. (1981). Another isomorphic theorem on fuzzy subgroups and fuzzy series of invariant subgroups. *Journal of Southwestern Petroleum Institute*, 1981. (In Chinese)

[10] Shi, Y. (1981). Convergence theorem of fuzzy integral of type II. *Journal of Southwestern Petroleum Institute*, 1981. (In Chinese)

[11] Sugeno, M. (1993). Fuzzy measure and fuzzy Integral. *Transactions of the Society of Instrument and Control Engineers*, 8, 2, 218-226, 1993.

[12] Sugeno, M., Terano, T. (1975). Analytical representation of fuzzy systems. *Special Interest Discussion on Fuzzy Automata and Decision Processes*, 1975.

[13] Terano, T., Sugeno, M. (1975). Conditional fuzzy measures and their applications. *Fuzzy Sets and Their Applications to Cognitive and Decision Processes* , 1975.

[14] Wu, W. (1981). Fuzzy normal subgroup. *Fuzzy mathematics*, 1, 21-30, 1981. (In Chinese).

[15] Xiong, Q. (1978) Universal algebra. *Shanghai Science and Technology Press*, 1978. (In Chinese)

[16] Zadeh, L. A. Fuzzy sets. *Information and control*, 8(3), 338-353, 1965.

[17] Zhang, W.; Zhao, R. (1980). Generalization of fuzzy measure and fuzzy integral *Science Report of Xi'an Jiaotong University*, 1980. (In Chinese)

[18] Zheng, D.; Jinli Huang, J.(1980). The neighborhood theorem of quasi-fixed points and the convergence theorem in measure of fuzzy integral *Modern Cybernetics Information*, 1980. (In Chinese)

[19] Zou, K. (1981). Fuzzy group theory. *Collection of Graduate Papers of Mathematics Department of Beijing Normal University*, 1981. (In Chinese)

Cite this paper as:
Shi, Y. (2021). My Early Researches on Fuzzy Set and Fuzzy Logic, *International Journal of Computers Communications & Control*, 16(1), 4090, 2021. https://doi.org/10.15837/ijccc.2021.1.4090