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Lyapunov-based Methods for Maximizing the Domain of Attraction

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Abstract

This paper investigates Lyapunov approaches to expand the domain of attraction (DA) of nonlinear autonomous models. These techniques had been examined for creating generic numerical procedures centred on the search of rational and quadratic Lyapunov functions. The outcomes are derived from all investigated methods: the method of estimation via Threshold Accepted Algorithm (TAA), the method of estimation via a Zubov technique and the method of estimation via a linear matrix inequality (LMI) optimization and genetic algorithms (GA). These methods are effective for a large group of nonlinear models, they have a significant ability of improvement of the attraction domain area and they are distinguished by an apparent propriety of direct application for compact and nonlinear models of high degree. The validity and the effectiveness of the examined techniques are established based on a simulation case analysis. The effectiveness of the presented methods is evaluated and discussed through the study of the renowned Van der Pol model.

Keywords: Lyapunov function, nonlinear model, asymptotic stability, equilibrium point, genetic algorithm, threshold accepted algorithm, LMI.

1 Introduction

During the previous decades, the complexity of approximating the attraction domain area has remained the focus of several benchmarking works [25, 30, 35, 36, 41], the references contained in them. The asymptotic stability area of physical systems is a significant property to be identified [18]. From a practical point of view, it is every so often insufficient to demonstrate the local asymptotic stability for point of equilibrium [4, 20, 21, 34, 42], however one may also require discerning the size of the stability region, as well [27, 33]. As a matter of fact, the stability area or attraction domain is described as the set of original starting criteria where the system states meet at the points of equilibrium [7, 8, 9]. Thus, it is vital to specify the state of this area. To achieve this, one can utilize a Lyapunov function [11, 13, 20, 25, 27, 32]. In fact, for a specified energy function providing stable local equilibrium, the biggest probable DA, the state of which is established by the Lyapunov energy function theory, is referred to be as the prevalent smooth function set integrated in a bounded domain where its derivative is negative [12, 22, 27, 40].

A significant number of approaches on estimating the area of attraction investigated in the literature are established exploiting the established statements of LaSalle and Lefschetz [26, 29, 31, 32]. These later are based upon appropriately selected Lyapunov functions in [23, 24, 27]. Considered as one of the original outcomes as regard as the problematic of attraction domains estimation, the Zubov's method has designed a Lyapunov function restricted by an open ball over a closed interval and consequently has sorted out an approximated DA [16, 43].

Similar complexity is detected when applying the algorithm synthesized by Knobloch and Kappel [8]. An LMI method to design polynomial Lyapunov functions for non-polynomial class of systems is provided in [10]. However, the problem of defining a truncation order for the Lyapunov function in the Taylor series expansion is not well addressed.

In [11, 15] authors designate an LMI based technique for approximating the so-titled robust DA for a polynomial class of systems with model polytope uncertainties.

In [9], authors formulate a fundamental analytical background in which particular classes of minimum based distance problems are resolved via LMI calculations [14]. The previous specified approaches, nevertheless, are appropriate for the class of smooth non-linear ODE systems.

In general, there is no generic techniques for determining Lyapunov functions as a group of nonlinear models [6, 8] are established. Nonetheless, Lyapunov theory is still considered as the most efficient method to analyze the nonlinear models stability, although the Lyapunov theorem does not need the class of algebraic functions that must be retained [1, 2, 3, 37]. In [11, 13] the maximum uncertain domain to preserve the stability and nominal performances (robustness) of the nonlinear optimal control systems, is estimated.

The goal of this paper is to recommend analytical methods motivating systematic approximating approaches of the DA. With the aim of getting a definite expression of the estimated DA, an investigation of the Lyapunov theory exploiting parameterized Lyapunov function is conducted as described in [6]. As the DA is related with a specified function of Lyapunov, the proposal entails selecting the optimal parameters to attain the most significant DA area. These later are designed as results to an optimization problem [5]. Output response designs with quantifiable premise parameters have also been employed in [22].

The paper structure is as follows. Section 2 provides a representation of the Carleman Linearization and the methodology of obtaining the Lyapunov functions based upon Lyapunov stability techniques and TAA. In Section 3, a recursive technique to calculate the DA through rational Lyapunov function is discussed. The fundamental principles of the GA are evaluated in the last part of this section. The assessment of the global accomplishment for the different techniques is performed by means of a simulation study carried out on the Van der Pol model. Section 4 focuses on the conclusions and future works.

2 Searching Lyapunov Function via Threshold Accepting Algorithms

2.1 The Carleman Linearization

The development of $f(x)$ into a Taylor series expansion provides [17, 19]:

$$f(x) = \sum_{l} A^l x^{[l]} \tag{1}$$

wherein:

$$A^l = \left\{ \frac{1}{l!} \frac{\partial^l f(0)}{\partial x_{i_1} \ldots \partial x_{i_l}} \right\} \tag{2}$$

and:

$$x^{[l]} = \underbrace{x \otimes x \otimes x \ldots \otimes x}_{l-Times} \tag{3}$$

denotes the l power Kronocker product of vector x Performing algebraic operations one can rewrite the vector derivative $\dot{x}^{[l]}$ as follows:

$$\dot{x}^{([l])} = \sum\nolimits_{j=1}^{l} x \otimes .. \otimes .. \underbrace{\dot{x}}_{j} \otimes .. \otimes x \tag{4}$$

This means

$$\dot{x}^{([l])} = \sum\nolimits_{j=1}^{l} x \otimes .. \otimes .. f(x) \otimes .. \otimes x \tag{5}$$

By substituting $f(x)$ with (1), one obtains:

$$\dot{x}^{[l]} = \sum\nolimits_{j=1}^{l} x \otimes .. \otimes .. \sum\nolimits_{k} A^k x^{[k]} \otimes .. \otimes x \tag{6}$$

Thus, it becomes

$$\dot{x}^{[k]} = \sum_{l} A_k^l x^{[k+l-1]} \tag{7}$$

where

$$A_k^l = A_1^l \otimes I^{[k-1]} + I \otimes A_{k-1}^l \tag{8}$$

Let the change of variables given below be considered now:

$$\Psi = \left(x, x^{[2]}, x^{[3]}, \ldots, x^{[k]}, \ldots \right)^T \tag{9}$$

an infinite dimension linear system given as:

$$\dot{\Psi} = H\,\Psi \tag{10}$$

is attained with:

$$H = \begin{bmatrix} H_1^1 & \ldots & H_1^i & \ldots & \ldots \\ 0 & H_2^1 & H_2^2 & \ldots & \ldots \\ 0 & 0 & H_3^1 & H_3^2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{11}$$

2.2 A Lyapunov theory case investigation

In this paragraph, an overall Lyapunov hypothesis for stabilizing a linear infinite dimension dynamical model is given. Consider the nonlinear system (1) where it is assumed that the origin is an equilibrium state (i.e.$f(0) = 0$). It is shown that system (1) can be expressed under a linear operator form with an infinite dimension as shown in (11) where the changed state vector is given by (10). Consider the nonlinear function given by [28]:

$$V = \left\langle \left(x, x^{[2]}, x^{[3]}, ... \right), \Upsilon \left(x, x^{[2]}, x^{[3]}, ... \right) \right\rangle \tag{12}$$

then

$$\begin{aligned} \dot{V} &= \left\langle \left(H^T \Upsilon + \Upsilon H \right) \left(x, x^{[2]}, .. \right) \left(x, x^{[2]}, .. \right) \right\rangle \\ &= - \left\| \left(x, x^{[2]}, x^{[3]}, ... \right) \right\|^2 \\ &= - \left(e^{\ \|x\|^2} - 1 \right) \end{aligned} \tag{13}$$

whereΥ is a symmetric positive definite operator and $\|.\|$ denotes the norm on the subset $\Im = \left\{ \left(x, x^{[2]}, ... \right), x \in \Re^n \right\}$ considered on the tensor algebra space $\Re^n$ over $\Re^n$. Consequently the defined function is a Lyapunov function and the studied system is globally asymptotically stable.

2.3 Main outcomes

Here, a finalized algorithm yielding a Lyapunov function for the class of investigated systems is presented. To achieve this aim, proposed functions of Lyapunov characterized by a quadratic formulation [39]:

$$V(x) = x^T \Upsilon x, \Upsilon = \Upsilon^T \tag{14}$$

are taken into account. A standard TAA [38] has been adjusted for sake of establishing the matrices and which satisfies the Lyapunov equation:$H^T \Upsilon + \Upsilon H = -\Theta$. The steps given below represent the synthesized technique for stability assessment.

– Initial step: Set zero matrixes Υ and Θ

– First step: Randomize matrix elements and validate the definite positive characteristic. IF Υ isn´ t positive definite, REDO first step.

– Second step: Calculate matrix Θ established in relation with Υ. Validate the definite negative propriety of Θ. If not satisfied, REDO first step.

– Final step: Approve final solutions.

2.4 Illustrative example

Consider the nonlinear system given as [43]:

$$\left\{ \begin{array}{l} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 + \left(x_1^2 - 1 \right) x_2 \end{array} \right. \tag{15}$$

The Carleman Linearization is used while the polynomial state equation is truncated to the third order. The equivalent linear system (11) is acquired as follows:

$$\Psi = \left[x_1, x_2, x_1^2, \ldots x_2^3 \right] \tag{16}$$

and

$$H = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 \end{bmatrix} \tag{17}$$

Applying the TAA, yields:

$$\Upsilon = \begin{bmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \end{bmatrix} \tag{18}$$

where:

$$\Upsilon_1 = \begin{bmatrix} 1.7894 & 0.6500 & -0.4242 \\ 0.6500 & 1.2914 & 0.0615 \\ -0.4242 & 0.0615 & 1.3590 \\ 0.0693 & -0.0081 & 0.3250 \\ 0.0146 & 0.2092 & 0.3114 \\ 0.0515 & -0.4560 & 0.2696 \\ 0.0520 & 0.0914 & 0.0695 \\ 0.0540 & 0.0325 & -0.1123 \\ 0.1635 & 0.0717 & -0.1578 \end{bmatrix}$$

$$\Upsilon_2 = \begin{bmatrix} 0.0693 & 0.0146 & 0.0515 \\ -0.0081 & 0.2092 & -0.4560 \\ 0.3250 & 0.3114 & 0.2696 \\ 0.6614 & 0.3636 & -0.0943 \\ 0.3636 & 0.6136 & -0.1231 \\ -0.0943 & -0.1231 & 2.1146 \\ 0.2623 & 0.0292 & 0.2667 \\ 0.1107 & 0.0571 & 0.1462 \\ 0.0732 & 0.0815 & 0.0435 \end{bmatrix}$$

$$\Upsilon_3 = \begin{bmatrix} 0.0520 & 0.0540 & 0.1635 \\ 0.0914 & 0.0325 & 0.0717 \\ 0.0695 & -0.1123 & -0.1570 \\ 0.2623 & 0.1107 & 0.0732 \\ 0.0292 & 0.0571 & 0.0815 \\ 0.2667 & 0.1462 & 0.0435 \\ 0.5518 & 0.2324 & 0.1160 \\ 0.2324 & 0.3183 & 0.1784 \\ 0.1160 & 0.1784 & 0.3950 \end{bmatrix}$$

and:

$$\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 & \Theta_3 \end{bmatrix}$$

$$\Theta_1 = \begin{bmatrix} -1.3000 & -0.1000 & -0.2001 \\ -0.1000 & -1.1000 & -0.4000 \\ -0.2001 & -0.4000 & -1.3000 \\ -0.5000 & 0.2001 & -0.6002 \\ -0.0998 & -0.6000 & -0.7000 \\ 0.3000 & 0.5000 & -0.0199 \\ -0.1999 & -0.1000 & -0.0999 \\ -0.2000 & 0.3000 & 0.3000 \\ -0.4002 & 0.0902 & -0.0099 \end{bmatrix}$$

$$\Theta_2 = \begin{bmatrix} -0.5000 & -0.0998 & 0.3000 \\ 0.2001 & -0.6000 & 0.5 \\ -0.6002 & -0.7000 & -0.0199 \\ -1.4000 & -0.0702 & -0.2999 \\ -0.0702 & -1.0000 & -0.0300 \\ -0.2999 & -0.0300 & -1.6002 \\ -0.8000 & 0.1997 & -0.0999 \\ -0.0501 & -0.0301 & -0.4997 \\ -0.2000 & -0.0898 & -0.0399 \end{bmatrix}$$

$$\Theta_3 = \begin{bmatrix} -0.1999 & -0.2000 & -0.4002 \\ -0.1000 & 0.3000 & 0.0902 \\ -0.0999 & 0.2992 & -0.0123 \\ -0.8000 & -0.0501 & -0.1992 \\ 0.1997 & -0.0301 & -0.0898 \\ -0.0999 & -0.4997 & -0.0399 \\ -1.4998 & -0.2000 & -0.0801 \\ -0.2000 & -0.7004 & -0.1001 \\ -0.0801 & -0.1001 & -1.2996 \end{bmatrix}$$

Therefore, the Lyapunov function $V(\Psi(x))$ and its first derivative $\dot{V}(\Psi(x))$ are expressed under in the forms $V(\Psi) = \Psi^T \Upsilon \Psi$ and $\dot{V}(\Psi) = \Psi^T \Theta \Psi$. The area plots respectively for the function $V(\Psi)$ and $\dot{V}(\Psi)$ are demonstrated in Figure 1 and Figure 2. Subsequently system (15) is globally asymptotically stable at origin.

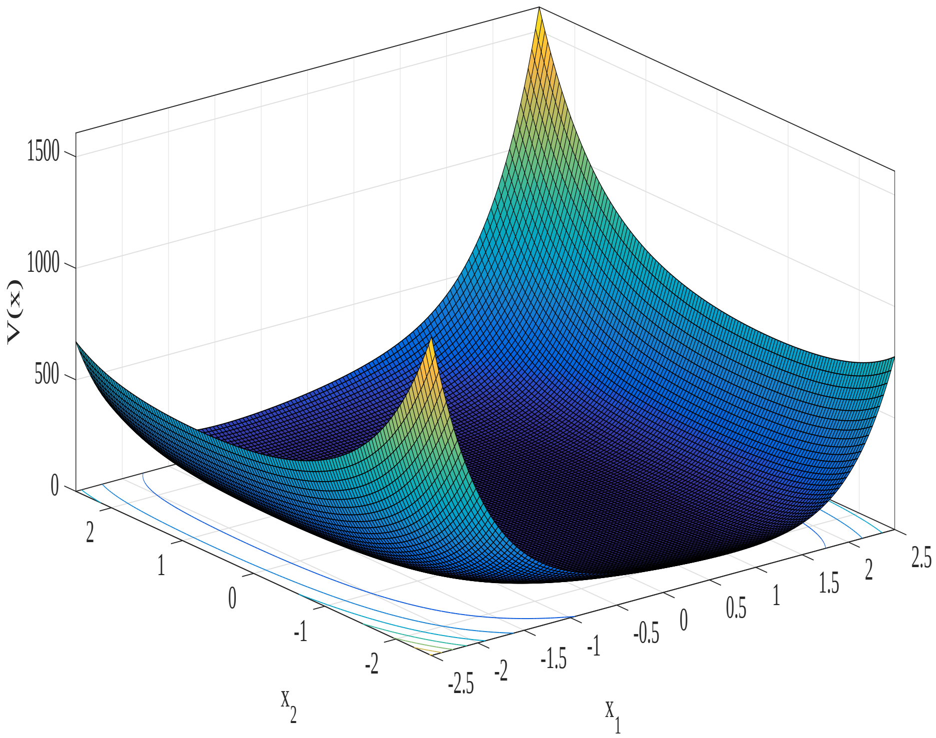

Figure 1: Area plots for the Lyapunov function $V(\Psi)$

2.5 Results and discussion

In this section, the investigation of the Carleman linearization letting the computation of an infinite dimensional linear model is provided. An analytical step procedure which defines a function of Lyapunov and ensures nonlinear models stability via the outlining of a Lyapunov equation is considered. This technique is based on the generic TAA [38]. The most important conclusion that is derived as a result of the implementation of this method on a second order system is given as follows:

Raising the truncation degree of the Lyapunov equation creates better accuracy on the typical DA. However, it is noteworthy that there is a quick increase in the linear matrix dimension. This causes difficulties in the calculation of the Lyapunov function with no appreciable enhancement in the DA

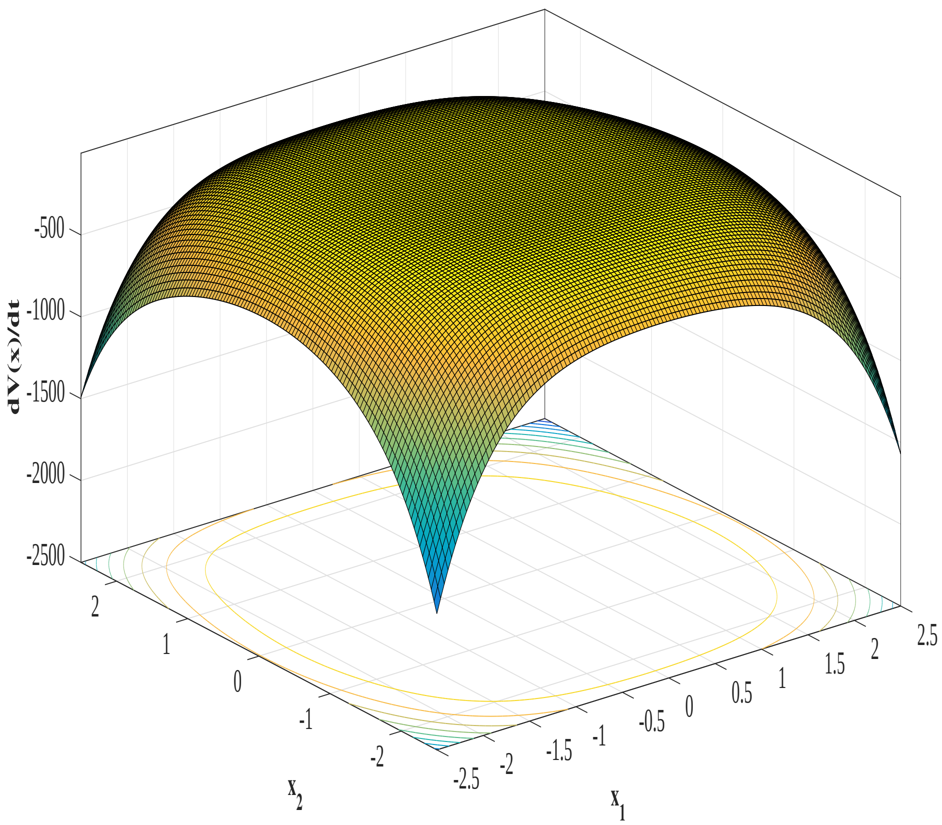

Figure 2: Area plots for the function $\dot{V}(\Psi)$

dimension. It would be an innovative technique to examine beforehand the developed technique to define the best possible degree of truncation for the highest accuracy in defining the DA.

3 Estimation of the DA using a Zubov approach

3.1 Rational Lyapunov function

This paragraph aims to explore an analytical approach for approximating the DA of nonlinear models. This later is established based on the calculation of a Lyapunov function providing a solution for the subsequent equation :

$$\dot{V}\left(x\right) = \left(\frac{dV\left(x\right)}{dx}\right)^{T} f\left(x\right) = -\phi\left(x\right) \tag{19}$$

With $\phi\left(x\right)$ a positive definite function. To solve the equation (19), a structured procedure using the conversion of $f(x)$ into Taylor series expansion was given in [43]. This procedure started with writing $f(x)$ under the following form:

$$f\left(x\right) = \sum_{i=1}^{\infty} F_i\left(x\right) \tag{20}$$

As the Lyapunov candidate function should be restricted when evolving towards the border (that's mean when $x \to \infty$), therefore, this attribute can be stated by considering a rational defined by [43] :

$$V\left(x\right) = \frac{N\left(x\right)}{D\left(x\right)} \tag{21}$$

where $N(x)$ and $D(x)$ are polynomial functions. Subsequently, $V(x)$ should be represented as follows:

$$V(x) = \frac{\sum\limits_{i=2}^{\infty} N_i(x)}{1 + \sum\limits_{i=1}^{\infty} D_i(x)} \tag{22}$$

Equation (20) and equation (22) yield:

$$(\kappa(D_i, N_i)) \sum_{i=1}^{\infty} F_i = -x^T I_n x (1 + \sum_{i=1}^{\infty} N_i)^2 \tag{23}$$

wherein:

$$\kappa(D_i, N_i) = \left(1 + \sum_{i=1}^{\infty} D_i\right) \sum_{i=2}^{\infty} \nabla N_i^T - \left(\sum_{i=1}^{\infty} \nabla D_i^T\right) \sum_{i=2}^{\infty} N_i \tag{24}$$

From this, finally one obtains:

$$\begin{aligned} &\sum_{i=2}^{\infty}\sum_{j=2}^{\infty} \nabla N_i^T F_j + \sum_{i=1}^{\infty}\sum_{j=2}^{\infty}\sum_{k=1}^{\infty} D_i \nabla N_j^T F_k - \sum_{i=1}^{\infty}\sum_{j=2}^{\infty}\sum_{k=1}^{\infty} \nabla D_i^T N_j F_k \\ &= -x^T I_n x \left(1 + 2\sum_{k=1}^{\infty} D_k + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} D_i D_j\right) \end{aligned} \tag{25}$$

Identifying the coefficient of equal degrees from both sides of the second order equation gives:

$$\nabla N_2^T F_1 = -x^T I_2 x \tag{26}$$

and:

$$F_1(x) = H_1 x = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{27}$$

When $k \geq 3$ the generic solution can be written as:

$$\begin{aligned} &\sum_{i=2}^{k} \nabla N_i^T F_{k+1-i} \\ &+ \sum_{i=1}^{j-2}\sum_{k=2}^{\infty} \left(D_i \nabla N_k^T - \nabla D_i^T N_k\right) F_{j+1-i-k} \\ &= -x^T Q_n x \left(1 + 2D_{k-2} + \sum_{i=1}^{k-3} D_i D_{k-2-i}\right) \end{aligned} \tag{28}$$

Hence, in every step of the synthesized algorithm, the below linearity under a set of established equations as the same form for equations (27) and (28) is obtained:

$$H_n q = b_n \tag{29}$$

where H_n is a matrix of an appropriate dimension given by:

$$H_n = \begin{bmatrix} H_{n1} & H_{n2} & B_n \end{bmatrix} \tag{30}$$

with:

$$H_{n1} = \begin{bmatrix} h_{11}n & h_{21} & 0 \\ h_{21}n & (h_{22} + h_{11}(n-1)) & 2h_{21} \\ 0 & 0 & h_{12}(n-1) \\ \vdots & \vdots & \vdots \\ 0 & 0 & \cdots \end{bmatrix} \tag{31}$$

$$H_{n2} = \begin{bmatrix} \cdots & \cdots & 0 \\ 0 & \cdots & 0 \\ [h_{22} + h_{11}(n-2)] & 3h_{21} & 0 \\ \cdots & 0 & 0 \\ 0 & h_{12} & nh_{22} \end{bmatrix} \tag{32}$$

B_n is obtained by expanding the quantity $\left(D_i \nabla N_2^T - \nabla D_i^T N_2\right) F_1$, b_n contains the numerical values of the equation (28).

$$q = [\, h_1, ..., \, h_{n+1} \, g_1 \, , \, ..., g_{n-1}]^T \tag{33}$$

Now the derivation of the Lyapunov function is as follows:

$$\dot{V}_n = -x^T Q_n x + \frac{e\,(x)}{\left(1 + \sum\limits_{i=1}^{n-2} Q_i\,(x)\right)^2} \tag{34}$$

Where $e(x)$ is a sum of monomials of degree greater than n. To enlarge the attraction area, in the proximity of the origin, one needs to analyze the condition $\dot{V}\,(x) < 0$ minimization process of the quantity $e\,(x)$. The problem thus implies the next reducing criterion:

$$\begin{cases} Min \; e_n\,(q) \\ \quad H_n q = b_n \end{cases} \tag{35}$$

with:

$$e_n\,(q) = \left\| \begin{matrix} \text{coefficients of the numerator terms} \\ \text{of (34)} \end{matrix} \right\|_2^2 \tag{36}$$

3.2 Study of an illustrative example

The earlier example of Van der Pol model is being investigated here. To identify the boarder of the DA, use a Lyapunov function defined by:

$$V_4\,(x) = \frac{\sum\limits_{i=2}^{4} R_i\,(x)}{1 + \sum\limits_{i=1}^{2} Q_i\,(x)} \tag{37}$$

with

$$R_i\,(x) = h_1 x_1^i + h_2 x_1^{i-1} x_2 + \cdots + h_i x_i^i$$

and

$$Q_i\,(x) = g_1 x_1^i + g_2 x_1^{i-1} x_2 + \cdots + g_i x_i^i$$

From equation (26) for degree 2, one can easily establish the equation of $R_2(x)$ described by:

$$R_2\,(x) = 1.5x_1^2 - x_1 x_2 + x_2^2 \tag{38}$$

For $k \geq 3$ the generic solution can be obtained based on the optimization method described by (35). The coefficients and are suggested to be coded into a 7 bits digital word as follows:

$$\begin{aligned} h_i &= (\; h_{00}, h_{01}, h_{02}, h_{03}, h_{04}, h_{05}, h_{06}) \\ g_i &= (g_{00}, g_{01}, g_{02}, g_{03}, g_{04}, g_{05}, g_{06}) \end{aligned} \tag{39}$$

The GA is employed where the genetic properties are summarized as follows: the size of population is 50, the rate of mutation is 0.1, the crossover rate is 0.65 and the maximum number for the generations is taken equal to 100 [24]. When implementing the recommended method based on the GA, the following outcomes are obtained:

$$V_4\,(x) = \frac{V_{n4}}{V_{D4}} = 3.18 \tag{40}$$

wherein:

$$\begin{aligned} V_{n4} = 1.5x_1^2 - x_1 x_2 + x_2^2 &- 0.2449x_1^4 + 0.4082x_1^3 x_2 \\ + 0.1582x_1^2 x_2^2 &- 0.1293x_1 x_2^3 - 0.0459x_2^4 \end{aligned}$$

$$V_{D4} = 1 + 0.2041x_1^2 + 0.0560x_1 x_2 - 0.0252x_2^2$$

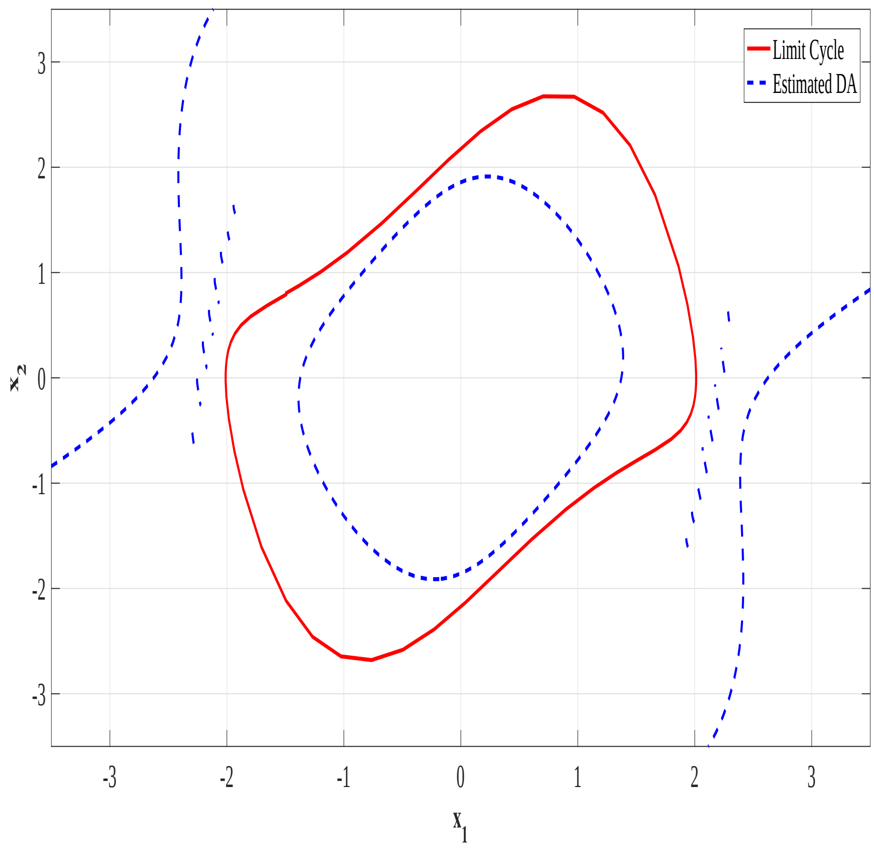

Figure 3: Stability region of Van der Pol system

3.3 Results and discussion

In this section, an algorithm leading to construct a rational Lyapunov function to approximate the DA of autonomous nonlinear systems is provided. Using this algorithm is advantageous compared to other one since there is no major complexity in selecting initial starting values. In fact, this technique is a linear programming based method which can be validated by an equivalence constrained minimization criterion. Moreover, the described technique can be of a practical use for a large set of nonlinear models. Indeed, it is not restricted for polynomial nonlinear systems which the case for the majority of existing algorithms. Equally important, the approximated DA is satisfactory because it is adequately large in comparison to the theoretical bound. However, the main disadvantages of the presented method can be mentioned as follows:

– The obtained DA has a non-definite geometric shape which means that it does not support a control strategy requiring the mathematical expression of the domain bound.

– The convergence of the developed algorithm is not demonstrated since it depends on a heuristic method (that is, the GA).

– The initial condition of the given algorithm impacts heavily the final outcome.

4 Conclusion

In this paper, the Lyapunov theory for approximating and the expanding of the domain of asymptotic stability of nonlinear models is examined. The investigated techniques have been performed with reference to numerical algorithms that were established using the integration of a rational Lyapunov function and a quadratic function. The primary motivation has been to study the various analytical approaches for the synthesis of DA in the proximity of an equilibrium point. The outcomes of the developments which are performed using the technique of threshold algorithms, the approximation method via a rational Lyapunov function and the approximation method based on LMI and GA demonstrated that this variety of approaches has the following attributes:

– They are effective for large classes of nonlinear models.

– They have the capability to enhance the domain of asymptotic stability.

– They can be performed for both high and low order nonlinear models.

– All the investigated techniques are developed in order to convert the DA estimation problem into a standard generic one.

The primary method known as the Carleman method demonstrated some difficulties which can be summed up as follows: when increasing the truncation order which enables the achievement of a Lyapunov function, an apparent complexity illustrates the digital operations of the dependent matrices describing the established Lyapunov function. The remaining paper discussed couple of approaches: the first is based on the result of the Zubov equation and the other is based on a statistical approach. To discover a representation of the Lyapunov function, it is demonstrated that solving the given problem corresponds to an optimization problem. A merging of the GA, LMI and RTM is employed in this respect. To analyse the performance of the various studied methods, a simulation analysis is accomplished based on the oscillator model of Van der Pol. This later is described by an asymptotic stability domain limited by a closed contour. The achieved outcomes have confirmed the effectiveness of the presented methods and approaches to ensure asymptotic stability in case of controlled systems. With the recommendations in this paper, conservativeness in the context of solutions from earlier literature was reduced: Solutions based on Lyapunov equations can be utilised as a seed for the techniques developed here.

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Author contributions

The authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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