INT J COMPUT COMMUN, ISSN 1841-9836
8(3):469-476, June, 2013.

On the Characteristic Functions of Fuzzy Systems

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Abstract: We provide several properties of the input-output function of the SISO / MISO Sugeno fuzzy systems with center of gravity defuzzification. The properties analyzed are related to continuity under various conditions for the input membership functions, including the case when the input space is a topological space.
Keywords: Sugeno fuzzy system , characteristic function, continuity, domain.

1 Introduction

The progress of knowledge on fuzzy systems historically followed a bifurcated path. On one way, the engineering applications drove the development [8] (Tong), with the underlying foundations following. On the second way, logic and mathematics produced advances with limited reference to the applications. That left several area in-between the two paths partly unclarified, especially regarding fuzzy logic systems (FLS) foundations.

We address several properties of the characteristic function of the SISO / MISO (Single Input Single Output / Multiple Input Single Output) fuzzy systems with defuzzification, for Sugeno fuzzy systems. As far as we know, these properties have been explicitly addressed until now only sporadically. The main issues concern the continuity of the characteristic functions of the fuzzy systems with center of gravity (c.o.g.) defuzzification, as continuity is essential in understanding the applicability and constraints when using fuzzy systems in control, system modeling (approximation), predictors and other current applications.

The approximation properties of fuzzy logic systems with defuzzification have been extensively studied starting in the 1990s [6] (Teodorescu 1990), [1] [2], [3], [7]. However, while somewhat elementary, the properties that guarantee that these systems have derivable or continuous input-output functions have not been investigated in detail, except cases discussed in [4], [9]. The continuity of FLSs is essential in many applications, and is effectively assumed in most control applications, as in [13] (even when not stated, e.g., as in many robotic system papers [10], [15]), and in interpolations and approximation with FLSs, as in [11], [12]. The topic was intensively studied, see e.g. [14], [9], [4] etc.

The organization of the paper is as follows. In the next section we recall a few definition and results and state several working hypotheses that are used in the paper. The third section addresses the continuity of Sugeno-type fuzzy systems. The last section is conclusive.

2 Definitions and hypotheses

Consider two spaces, $S \subseteq X$ named input space and $S\prime \subseteq Y$ named output space, and applications $\mu : S \to [0, 1] \subset \mathbf{R}$ named input membership functions and respectively $\eta\dot{:}S\prime \to [0, 1] \subset \mathbf{R}$ named output membership functions. The definition domain of all input membership functions is S; the subset of S where the membership functions have values larger than 0 is named the support of the function.

We work under a set of very general hypotheses that simplify the presentation of the main results; most results remain valid without these hypotheses, with slightly more elaborate conditions, but their proofs become tedious and lengthy. The hypotheses are as follows.

All input and output membership functions (H1) are normalized, that is, there is at least one point where the membership functions have value 1.

(H2) have a finite set of α-intervals for every α. (We will not use H2 when building an example in Section 3.)

(H3) have compact supports, that is, compact sets where they are larger than zero.

(H4) have bounded supports.

(H5) The union of all the supports of the input membership functions is compact. This hypothesis has no significant impact on the results, but simplifies the discussion.

Hypothesis H3 is typically accepted for fuzzy sets, yet the hypothesis can be removed by defining two (or several) fuzzy sets with compact support, with their union equal to the given non-compact support set. We will still require that the number of disjoint sets of the support is finite.

Hypothesis H4 (boundedness of the support) is only used to avoid studying one more trivial case for membership functions. It can be removed with no influence on the results. Hypothesis H5 (compactness of the union of the supports) plays no role except of avoiding splitting the problem into sub-problems.

Notice that in many respects, the type of input space is not essential, as far as it is a topological space where continuity of the mappings can be defined. In general, we assume that an input membership function is an application from a compact set in a Hausdorff topological space to the unit interval of the real line, $\mu : S \subseteq X \to [0,1] \subseteq \mathbf{R}$. We assume that the input membership functions are not singletons on X.

Consider a finite set of real numbers, named singletons, $\{\beta_i\}_{i=1,..,Q}$. We recall a few definitions.

Definition 1. A SISO zero-order Sugeno fuzzy system is a construction comprising (i) a set of input membership functions $\mu_k : S \to [0,1], k = 1,\cdots,m$, (ii) a set of output singletons, $\{\beta_h\}$; (iii) an application $i(.) : \{1,a,m\} \to \{1,\cdots,Q\}$, that associates to any $k \in 1,\cdots,m$ a value $i(k) \in \{1,\cdots,Q\}$; each such individual association, that is a value i(k) for a specified k, is named rule; (iv) an application $S \to \mathbf{R}$ defined by $x \to y = \frac{\sum_k \beta_{i(k)}\mu_k(x)}{\sum_k \mu_k(x)}, \beta_{i(k)} \in R$.

The part (iii) in the construction is equivalent to saying that there is a set of m rules, $R(k)$: If input is μ_k, then output is $\beta_{i(k)}$, that is, one singleton is assigned to each input membership function.

A definition a little more general allows for several singletons assigned to one input membership function,

(iii*) An application $i(.) : \{1,\cdots,m\} \to \wp\{1,\cdots,Q\}$, where $\wp$ denotes the set of parts (power set); $i(k)$ for a specified k is named *rule*; (iv) an application $S \to \mathbf{R}$ defined by

$x \to y = \frac{\sum_k \beta_h \max_{js.t.i(j)=h} \mu_j(x)}{\sum_k \max_{js.t.i(j)=h} \mu_j(x)}$.

The part (iii*) in the construction is equivalent to saying that there is a set of rules as

$R(k)$: If input is μ_k, then output is $\beta_{i1(k)}, \beta_{i2(k)}, \cdots, \beta_{ir(k)}$,

that is, several singletons may be assigned simultaneously to a single input (non-univalent, multivalued / multivocal association). Equivalently, one can allow for several rules with the same antecedent and different consequents connected by OR, as

$R(k)$: If input is μ_k, then output is $\beta_{i1(k)}$, OR

$R(k)$: If input is μ_k, then output is $\beta_{ir(k)}$.

The application $x \in S \subseteq X \to y \in Y \subseteq \mathbf{R}$ is named characteristic or input-output function of the FLS. This function is defined in all points where at least one input membership function is not null, for Sugeno-type FLSs. This condition is required by the c.o.g. defuzzification method, specifically by the condition that the denominator in the c.o.g. is not null. For ease, we will say that a fuzzy system is continuous if its characteristic function is continuous.

3 Properties of continuity SISO Sugeno systems

The simplest way to analyze the continuity of Sugeno-type system is to use the formula of the output according to the definition in the previous Section and to consider that all input membership functions are defined on the whole domain, X. When an input membership function is defined on a subset of the input space, $\mu : S \subseteq X \to [0,1]$, the last condition is enforced by extending the definition of the input membership functions to the whole X by $\mu(x \in X \setminus S) = 0$. We will study the cases when all input membership functions are continuous on X. Because we assume that all membership functions are continuous everywhere, moreover because the maximum, minimum and rational functions of continuous functions are continuous (wherever they are defined), we directly have the basic result:

Proposition 1. Sugeno SISO fuzzy systems with continuous membership functions over the whole input space are continuous input-output applications in the subset of the input space where the system is defined.

The input space can be whatever topological space X that allows us building continuous applications from it to $[0,1]$. Beyond this very general condition, all the other conditions regard the interval $[0,1]$ and functions defined on it; hence, the generality of the result in Proposition 1.

While the above result is very general, easy to obtain, in addition easily extendable to all Sugeno systems (of whatever order), it sheds little light on the internal mechanisms of these systems and on what happens outside the subset of the input space where the system is defined. Therefore, we will provide another proof for the continuity, studying along the proof the properties of the frontier of the definition space of the system. The analysis will be developed first for the simplest case of FLSs, namely when the input membership functions are defined on $\mathbf{R}$. Then, we deal with X as a general topological space.

We first introduce a few concepts that simplify the explanations. Consider single-input single-output (SISO) fuzzy systems with input membership functions defined on the real axis (mono-dimensional input functions). We assume that any input membership function is not null on an opened interval and null outside it. Denote by $I_k = (\underline{x}_k, \overline{x}_k)$ the opened interval corresponding to the membership function μ_k, with $\mu_k(x) > 0$ for all $x \in I_k$ and $\mu_k(x) = 0$ for all $x \in \mathbf{R} \setminus I_k$. The case $\underline{x}_k = \overline{x}_k$ is not allowed, as the membership function would be void everywhere except a point, reducing it to an input singleton. While not essential, for ease of exposition consider that the union of all closures $\overline{I}_k$ of the intervals I_k is an interval,$\overline{I} = \cup_k \overline{I}_k$. Consider that the intersection of the closures of two specified intervals, $\overline{I}_k$ and $\overline{I}_j$, is not void. The following cases will play a role in the discussion.

The intersection of two intervals is a single point. Assume that $\overline{I}_k \cap \overline{I}_j = \overline{x}_k = \underline{x}_j$ or $\overline{I}_k \cap \overline{I}_j = \overline{x}_j = \underline{x}_k$. The two intervals are adjacent, non-overlapping. (See Fig. 1 a). Then, in the respective point $\underline{x}_k = \overline{x}_j$ or $\underline{x}_j = \overline{x}_k$, both functions have zero value, therefore another membership function must have non-zero value in that point, for the fuzzy system is defined in that point. In the respective point (x_3 in Fig. 1 (a)), the system is not defined. If the two membership functions are associated with different singletons, the system will have different values at left and at right of x_3, making it impossible to achieve continuity of the system simply by extending the characteristic function of the system to one of the two values.

The cases $\overline{I}_k \cap \overline{I}_j = \underline{x}_k = \underline{x}_j$ and $\overline{I}_k \cap \overline{I}_j = \overline{x}_k = \overline{x}_j$ are not possible, because we required that $\underline{x}_k \neq \overline{x}_k$ and that the intersection reduces to a single point. Therefore, this case is not allowed under normal circumstances (Sugeno systems are not using input singletons).

The intersection is an interval. When $\overline{I}_k \cap \overline{I}_j = \overline{I}_k$ or $\overline{I}_k \cap \overline{I}_j = \overline{I}_j$, the membership functions may look as in Fig. 1 b. We will take into account this case. The typical case in applications is an intersection that is an interval but not equal to one of the two, I_k, I_j.

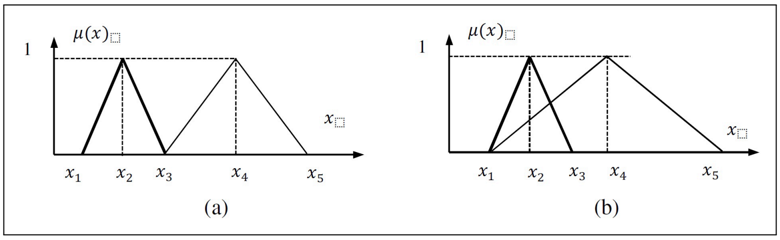

Figure 1: (a) not allowed case, if no other function is non-zero in x_3; (b) allowed case.

Lemma 1. In the case (ii) above, when $I_k \neq I_j$, there is at least one point inside $\overline{I}_k \cap \overline{I}_j$ such that the two membership functions have equal values.

Proof. Indeed, as the membership functions are assumed continuous, so is their difference, $\mu_k - \mu_j$. Denote $\overline{I}_k \cap \overline{I}_j = [a, b]$. Then, either $(\mu_k - \mu_j)(a) < 0$ and $(\mu_k - \mu_j)(b) > 0$, or $(\mu_k - \mu_j)(a) > 0$ and $(\mu_k - \mu_j)(b) < 0$. Therefore, there is a point $x \in [a, b]$ such that $(\mu_k - \mu_j)(x) = 0$.

Consider all points where two membership functions have equal values. Denote these points by e_k and create the ordered set comprising all distinct $e_k, \underline{x}_k, \overline{x}_k$ points. Denote this set of ordered points by $P_C = a_h$. The set of intervals $[a_o, a_1), \cdots, [a_h, a_{h+1}), \cdots, [a_N, a_{N+1}]$ will be named canonical partition of the interval I. The points e_k play a role in considerations only regarding the derivability of Sugeno systems and regarding Mamdani systems; for sake of generality we use them here too. One could define the canonical partition without the points e_k and introduce them only when needed.

Lemma 2. For any interval in the canonical partition, a single rule (iii) or a single subset of rules (iii*) simultaneously apply (are active) in a single-input fuzzy logic system.

Proof. Consider x changing inside some interval of the canonic partition. Because such an interval is included or at most equal to any support (basis interval) for a membership function, moreover no point inside such an interval, $x \in (a_h, a_{h+1})$, represents an edge of the support of an input membership function, the activated rules will remain the same for all inside points.

We recall that the minimum and maximum of two continuous functions are continuous functions, that is, if f and g are continuous functions in some interval, $\min(f(x), g(x))$ and $\max(f(x), g(x))$ are continuous functions on that interval. Consequently, the truncation operation applied to a continuous membership function μ, $\min(\mu(x), \gamma)$, as used for Mamdani systems, is continuous for any $\gamma \in \mathbf{R}$, in the given interval of continuity of μ. We introduce the notion of support of a fuzzy systems defined by the union of the supports of all the input membership functions, $I = \cup_k I_k$. We now can prove

Proposition 2. Any SISO zero-order Sugeno system with continuous input membership functions defined on $\mathbf{R}$ is continuous inside the support of the system.

Proof. According to Lemma 2, a single subsystem of rules always apply inside an interval of the canonical partition, thus the set of integers k for which $\mu_k(x) \neq 0$ in $x \to y = \frac{\sum_k \beta_{i(k)} \mu_k(x)}{\sum_k \mu_k(x)}$

is uniquely defined. As the functions $\mu_k(x)$ are continuous and none null, the denominator is nowhere null inside the interval and both the nominator and denominator are continuous, thus the function $x \in I \to y \in \mathbf{R}$ is continuous.

In the points ending the interval for x, $x \in (a_h, a_{h+1})$, the subset of active rules changes because one or several membership functions become zero. Consider the point a_h, which is not on the frontier of the support of the fuzzy system, $a_h \in \overline{I} \setminus I$. At the limit, we have

$$\lim_{x \to a_h, x > a_h} \frac{\sum_k \beta_{i(k)} \mu_k(x)}{\sum_k \mu_k(x)}.$$

Some of the membership functions tend to zero, while others tend to some non-zero values. Those membership functions not tending to zero remain non-null for $x < a_h$ in some vicinity of a_h, because of continuity. At the same time, other membership functions are non-zero for $x < a_h$ and tend to zero in a_h. Therefore, the limit at right and at left of a_h are equal,

$$\lim_{x \to a_h, x > a_h} \frac{\sum_k \beta_{i(k)} \mu_k(x)}{\sum_k \mu_k(x)} = \lim_{x \to a_h, x < a_h} \frac{\sum_k \beta_{i(k)} \mu_k(x)}{\sum_k \mu_k(x)}$$

and the application is continuous in a_h. Because in the whole closed interval $[a_h, a_{h+1}]$ the system is continuous, it is continuous in the whole definition domain.

The only case we still need to analyze is on the frontier of the union of the supports. Because all the subsequent considerations do not make use of the specific fact that the input membership functions are defined on R, we will continue the discussion in a more general setting. Denote $U = \cup_k S_k$, where S_k is the support of the membership function μ_k. The frontier is $\Gamma = \overline{U} \setminus U$. Because on the frontier all membership functions are null, the FLS is not defined on Γ , yet it can be extended on Γ if the system function has a limit on every point on Γ.

$$\lim_{x \to x_0 \in \Gamma, x \in U} \frac{\sum_k \beta_(i(k)) \mu_k(x)}{\sum_k \mu_k(x)} = \lim_{x \to x_0 \in \Gamma, x \in U} \frac{\frac{\sum_k \beta_{i(k)} \mu_k(x)}{\min_k \mu_k(x)}}{\frac{\sum_k \mu_k(x)}{\min_k \mu_k(x)}}$$

.

In the second limit, the denominator is larger than 1, because all quantities are positive and for $h = \arg\min_k \mu_k(x)$, $(\mu_h(x))/\min_k \mu_k(x) = 1$. When a single membership function has values larger than zero in a vicinity of x_0, that is, when $x \in V_{x_0} \cap U\{x_0\}$ implies that there is some unique h such that $\mu_h(x) > 0$ and $\mu_{k \neq h}(x) = 0$, then the limit is $\beta_{i(h)}$. This is probably the most frequent case in applications.

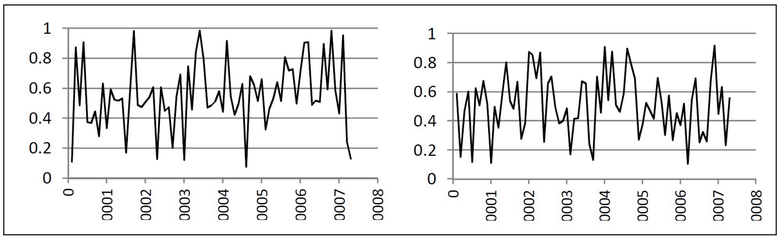

Figure 2: Evolution near zero of the system described in Example 1

However, in general the limit does not exist.

Example 1. Consider $\mu_1(x) = \max(1, x|sin(1/x)|)$, $\mu_2(x) = \max(1, x|sin(1/(2x))|), \beta_(i(1)) = 1, \beta_i(2) = 3$. Notice that these functions do not obey the requirement in the hypothesis (H2). The limit

$$\lim_{x\to 0, x>0} \frac{(\max(1, x|sin(1/x)|) + \sqrt{3}\max(1, x|sin(1/\sqrt{2}x)|)}{\max(1, x|sin(1/x)|) + \max(1, x|sin(1/\sqrt{2}x)|)}$$

does not exist. However, this is not a case of practical interest when choosing membership functions for typical control problems. Figure 3 shows the variation of the above function around zero, at two scales. On the other hand,if for any point x on the frontier of the support of the fuzzy system with continuous input membership functions on X, there is a membership function μ_h such that for all the other membership functions $\mu_{i\neq h}$ there is a finite limit

$$\lim_{u\to x} \frac{\mu_i(u)}{\mu_h(u)} = L_{i,h(x)}(x),$$

then the characteristic function of the Sugeno system can be extended on the frontier as

$$x \to y = \frac{\sum_k \beta_{i(k)}\mu_k(x)}{\sum_k \mu_k(x)}$$

for x inside S^*

$$x \to y = \frac{\beta_(i(h(x))) + \sum_(k \neq h(x))L_(i, h(x))(x))}{1 + \sum_(k \neq h(x))L_(i, h(x))(x)}, x \in \Gamma(S^*)).$$

The extension is obtained as $\lim_{u\to x} \frac{\sum_k \beta_(i(k))\mu_k(x)}{\sum_k \mu_k(x)} = \beta_{i(h(x))} + \sum_{k=h(x)} \beta_{i(k)}\mu_k(x))/(1 + \sum_{k=h(x)} L_{i,h(x)}(x)$. The limit is obtained by factoring in the nominator and in the denominator by $\mu_h(u)$. The possibility to choose several membership functions for μ_h does not modifies the result, as it is easy to check. Example 1. Consider a Sugeno system with only two membership functions, both defined on $[0, 0.1]$, $\mu_1(x) = x, \mu_2(x) = x^2, \beta_1 = 1, \beta_1 = 3$. In this case, $h = 1$. Then, inside $[0, 0.1], x \to y = (x+3x^2)/(x+x^2)$, thus $\lim_{x\to 0}(y) = \lim_{x\to 0}(x+3x^2)/(x+x^2) = 1$.

Remark 1. The characteristic function of the system so defined is continuous in the closure of the support, S^* if the input membership functions are continuous in X. This property results from the above discussion.

Remark 2. Consider the more general case when X is a multidimensional space, for example $x \in X$ and $X = \mathbf{R}^2$ and $\Gamma(S^*)$ is a curve in the plane. It is easy to see that the function $x \to y = (\beta_{i(h(x))} + \sum_{k\neq(x)} L_{i,h(x)}(x)/(1 + \sum_{k\neq h(x)} L_{i,h(x)}(x)$ defined on $\Gamma(S^*)$ may be discontinuous if there are membership functions that are discontinuous even in only one of the variables of the plane.

Because the standard definition of the fuzzy logic systems forces us to exclude the definition of the characteristic function on the support of the system, that is, forces the support to be an open set, we suggest the next extension of the definition for SISO Sugeno systems by replacing condition (iii) in the definition:

(iii**) The output of the system is

$$x \to y = \lim_{u\to x} \sum_k \beta_{i(k)}\mu_k(u) / \sum_k \mu_k(u),$$

whenever the limit exists. This definition generalizes (iii) covering it as a specific case. In addition, it allows us to use membership functions as in Fig. 1 (b) and still have the system defined in the closed interval corresponding to the bases of the triangles. Moreover, the definition completes the domain of definition of the system to a closed set, and makes it continuous (when the input membership functions are continuous and satisfy the other conditions regarding the limit) on its whole (closed) domain. Of course, outside the reunion of the input membership functions support, the system is undefined.

The case of multi-input single-output MISO case, with membership functions defined on R for each input (mono-dimensional input functions) is dealt similarly. We do not study it here because of space limits.

4 Conclusions

We provided several properties of the input-output (characteristic) function of the SISO and MISO Sugeno fuzzy systems with defuzzified output by the center of gravity method. These properties regard the continuity of the characteristic functions of the fuzzy systems and are essential in understanding the applicability and constraints when using fuzzy systems in control, system modeling (approximation), predictors and other current applications. We made a clear difference between the domain of definition of the input membership functions and the domain of the characteristic (input-output) function of the fuzzy system, stressing that they are not identical. The continuity inside the system domain was proved for input topological spaces that allow the definition of the membership functions. Next, we showed cases when it is possible to and explained how to extend the definition of Sugeno fuzzy systems to a closed domain, preserving the continuity of the system. While we provided a proof of continuity for the opened domain of the systems, proof that is valid for general topologic spaces, we worked out a second proof, which is longer but has the advantage of detailing the internal mechanics of Sugeno systems. The second way of dealing with the continuity problem has the additional advantage of showing a path toward the extending of the domain of the systems to a closed domain, for a case that may be of interest in applications.

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