

Fuzzy Szpilrajn Theorem through Indicators

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Abstract: In this paper there are studied some numerical indicators which measure the degree to which a fuzzy relation verifies some properties (reflexivity, transitivity, etc.). The main result is a fuzzy generalization of the Szpilrajn theorem in terms of such numerical indicators and is applied to any fuzzy relation.

Keywords: fuzzy relation, Szpilrajn theorem, similarity

1 Introduction

The classical Szpilrajn theorem [10] asserts that any partial order can be extended to a total order. This result has been followed by refinements and generalizations and has been used in applications too (e. g. consumer theory [8]).

The first fuzzy version of the Szpilrajn theorem has been established by Zadeh [11]. Other fuzzy versions of this theorem can be found in [4], [6]. In [3] the topic is systematically studied in the framework of the fuzzy orders with respect to a left-continuous t -norm $*$ and a $*$ -similarity relation Ω .

The idea of this paper is the following: instead of studying a property P of a fuzzy relation R (e. g. reflexivity, transitivity, etc.) to define numerical indicators which should express "the degree to which the fuzzy relation R verifies the property P ". In this way, instead of considering a fuzzy order R on a set X we will have a number $Ord(R)$ which should measure "the degree to which R is a fuzzy order".

The main result of the paper is a generalization of the Szpilrajn theorem expressed in terms of such numerical indicators. It is a refinement of Theorem 6.2 in [3] and it is applied to any fuzzy relation.

2 Preliminaries

In this section we shall recall some basic facts on the residuum associated with a left-continuous t -norm and on fuzzy relations ([1], [2], [4], [5], [7], [9]).

For any $a, b \in [0, 1]$ we denote $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. More generally, for any set $\{a_i\}_{i \in I} \subseteq [0, 1]$ we denote $\bigvee_{i \in I} a_i = \sup\{a_i | i \in I\}$ and $\bigwedge_{i \in I} a_i = \inf\{a_i | i \in I\}$.

Let $*$ be a left-continuous t -norm [7], [5]. The *residuum* \rightarrow associated with $*$ is introduced by $a \rightarrow b = \bigvee\{c \in [0, 1] | a * c \leq b\}$.

The *biresiduum* \leftrightarrow is denoted by $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$.

We fix a left-continuous t -norm $*$.

Lemma 1. [1], [5] For any $a, b, c \in [0, 1]$ the following properties hold:

(1) $a * b \leq c$ iff $a \leq b \rightarrow c$;

(2) $a \wedge b = a * (a \rightarrow b)$;

(3) $a \leq b$ iff $a \rightarrow b = 1$;

(4) $a = 1 \rightarrow a$;

(5) $1 = a \rightarrow a$;

(6) $a \leq a * (a \leftrightarrow b)$.

Lemma 2. [1], [5] For any $\{a_i\}_{i \in I} \subseteq [0, 1]$ and $a \in [0, 1]$ the following properties hold:

$$(1) \left(\bigvee_{i \in I} a_i\right) * a = \bigvee_{i \in I} (a_i * a);$$

$$(2) a \rightarrow \left(\bigwedge_{i \in I} a_i\right) = \bigwedge_{i \in I} (a \rightarrow a_i);$$

$$(3) \left(\bigvee_{i \in I} a_i\right) \rightarrow a = \bigwedge_{i \in I} (a_i \rightarrow a).$$

Let X be a non-empty subset. A *fuzzy subset* of X is a function $A : X \rightarrow [0, 1]$. Denote by $\mathcal{F}(X)$ the family of the fuzzy subsets of X . For any $A, B \in \mathcal{F}(X)$ denote $A \subseteq B$ if $A(x) \leq B(x)$ for any $x \in X$.

A *fuzzy relation* on X is a function $R : X^2 \rightarrow [0, 1]$. R is said to be

- reflexive if $R(x, x) = 1$ for any $x \in X$;
- symmetric if $R(x, y) = R(y, x)$ for any $x, y \in X$;
- $*$ -transitive if $R(x, y) * R(y, z) \leq R(x, z)$ for all $x, y, z \in X$;
- strongly complete if $R(x, y) \vee R(y, x) = 1$ for any $x, y \in X$.

A reflexive, symmetric and $*$ -transitive fuzzy relation Ω on X will be called $*$ -similarity relation.

Let Ω be a $*$ -similarity relation on X and R a fuzzy relation on X . R is said to be:

- Ω -reflexive if $\Omega(x, y) \leq R(x, y)$ for any $x, y \in X$;
- $(*, \Omega)$ -antisymmetric if $R(x, y) * R(y, x) \leq \Omega(x, y)$ for any $x, y \in X$.

A $(*, \Omega)$ -order is $*$ -transitive, Ω -reflexive and $(*, \Omega)$ -antisymmetric fuzzy relation R on X . Let R, Q be two $(*, \Omega)$ -orders on X . We say that Q is an *extension* of R if $R \subseteq Q$, i. e. $R(x, y) \leq Q(x, y)$ for all $x, y \in X$.

The following fuzzy generalization of the Szpilrajn theorem was proved in [3]:

Theorem 3. Let Ω be a $*$ -similarity relation on X . Then any $(*, \Omega)$ -order on X has a strongly complete extension.

3 Some indicators

Let $*$ be a left-continuous t-norm and Ω be a $*$ -similarity relation on X .

Definition 4. For any fuzzy relation R on X let us define:

$$Ref(R) = \bigwedge_{x \in X} R(x, x);$$

$$Trans(R) = \bigwedge_{x, y, z \in X} [R(x, y) * R(y, z) \rightarrow R(x, z)];$$

$$Ref_{\Omega}(R) = \bigwedge_{x, y \in X} (\Omega(x, y) \rightarrow R(x, y));$$

$$Ant_{\Omega}(R) = \bigwedge_{x, y \in X} [R(x, y) * R(y, x) \rightarrow \Omega(x, y)];$$

$$SC(R) = \bigwedge_{x, y \in X} (R(x, y) \vee R(y, x));$$

$$Ord_{\Omega}(R) = Ref_{\Omega}(R) \wedge Ant_{\Omega}(R) \wedge Trans(R).$$

Lemma 5. For any fuzzy relation R the following equivalences hold:

$$(1) Ref(R) = 1 \text{ iff } R \text{ is reflexive};$$

$$(2) Trans(R) = 1 \text{ iff } R \text{ is } * \text{-reflexive};$$

- (3) $Ref_{\Omega}(R) = 1$ iff R is Ω -reflexive;
- (4) $Ant_{\Omega}(R) = 1$ iff R is $(*, \Omega)$ -antisymmetric;
- (5) $SC(R) = 1$ iff R is strongly complete;
- (6) $Ord_{\Omega}(R) = 1$ iff R is a $(*, \Omega)$ -order.

$Ref(R)$ will be called the degree of reflexivity of R , $Trans(R)$ the degree of $*$ -transitivity of R , etc. The indicators introduced above refine the properties of reflexivity, transitivity, etc. of fuzzy relations. Thus, instead of saying that the fuzzy relation R is reflexive, the real number $Ref(R)$ will measure "the degree to which R is reflexive".

Proposition 6. *Let R be a fuzzy relation on X and $x, y, z \in X$. Then*

- (1) $Ref(R) \leq R(x, x)$;
- (2) $Trans(R) * R(x, y) * R(y, z) \leq R(x, z)$;
- (3) $Ref_{\Omega}(R) * \Omega(x, y) \leq R(x, y)$;
- (4) $Ant_{\Omega}(R) * R(x, y) * R(y, x) \leq \Omega(x, y)$;
- (5) $Ord_{\Omega}(R) \leq \Omega(x, y) \leftrightarrow (R(x, y) * R(y, x))$;
- (6) $Ord_{\Omega}(R) \leq R(x, x)$.

4 Main result

In this section we shall prove a generalization of the theorem of Szpilrajn formulated in terms of the indicators introduced in the previous paragraph. The result will be valid for any fuzzy relations and in particular one will obtain Theorem 3.

Let $*$ be a left-continuous t-norm and Ω a $*$ -similarity relation on X . If R and Q are two fuzzy relations on X then we denote

$$R \preceq Q \text{ iff } R \subseteq Q \text{ and } Ord_{\Omega}(R) \leq Ord_{\Omega}(Q).$$

It is easy to see that \preceq is a partial order on the set of the fuzzy relations defined on X . If Q is a fuzzy relation on X then we denote by $Ext(Q)$ the set of all fuzzy relations R on X with the property that $Q \preceq R$.

Lemma 7. *If Q is a fuzzy relation on X then the partially ordered set $(Ext(Q), \preceq)$ admits a maximal element.*

Proof. We prove that $(Ext(Q), \preceq)$ is inductive. We consider a chain $(R_i)_{i \in I}$ in $Ext(Q)$: for any $i, j \in I$ we have $R_i \preceq R_j$ or $R_j \preceq R_i$. Of course $Q \preceq R_i$ for any $i \in I$. We will denote $R = \bigcup_{i \in I} R_i$. It suffices to prove that $R \in Ext(Q)$. It is obvious that $Q \subseteq R$ therefore we have to prove that $Ord_{\Omega}(Q) \leq Ord_{\Omega}(R)$. We show first that

$$(a) \quad Ord_{\Omega}(Q) \leq Ref_{\Omega}(R).$$

Let $x, y \in X$ and $i \in I$. Since $Q \subseteq R_i$ it follows immediately

$$Ord_{\Omega}(Q) \leq Ref_{\Omega}(Q) \leq Ref_{\Omega}(R_i) \leq \Omega(x, y) \rightarrow R_i(x, y).$$

By applying Lemma 1 (2) and the previous inequality

$$\begin{aligned} Ord_{\Omega}(Q) * \Omega(x, y) &\leq \Omega(x, y) * (\Omega(x, y) \rightarrow R_i(x, y)) = \\ &= \Omega(x, y) \wedge R_i(x, y) \leq R_i(x, y) \leq R(x, y) \end{aligned}$$

from where according to Lemma 1 (1), $Ord_{\Omega}(Q) \leq \Omega(x, y) \rightarrow R(x, y)$. It follows

$$Ord_{\Omega}(Q) \leq \bigwedge_{x,y \in X} (\Omega(x,y) \rightarrow R(x,y)) = Ref_{\Omega}(R).$$

We intend to prove that

(b) $Ord_{\Omega}(Q) \leq Ant_{\Omega}(R)$

Let $x, y \in X$. Then

$$\begin{aligned} Ord_{\Omega}(Q) * R(x,y) * R(y,x) &= Ord_{\Omega}(Q) * [\bigvee_{i \in I} R_i(x,y)] * [\bigvee_{j \in I} R_j(y,x)] = \\ &= \bigvee_{i,j \in I} Ord_{\Omega}(Q) * R_i(x,y) * R_j(y,x). \end{aligned}$$

Let $i, j \in I$. Assume that $R_i \preceq R_j$ therefore $R_i \subseteq R_j$ and $Ord_{\Omega}(R_i) \leq Ord_{\Omega}(R_j)$. Then, according to $Q \preceq R_j$ and Proposition 6 (4):

$$\begin{aligned} Ord_{\Omega}(Q) * R_i(x,y) * R_j(y,x) &\leq Ord_{\Omega}(R_j) * R_j(x,y) * R_j(y,x) \leq \\ &\leq Ant_{\Omega}(R_j) * R_j(x,y) * R_j(y,x) \leq \Omega(x,y). \end{aligned}$$

Since this inequality is valid for any $i, j \in I$ it follows $Ord_{\Omega}(Q) * R(x,y) * R(y,x) \leq \Omega(x,y)$, therefore, according to Lemma 1 (1), $Ord_{\Omega}(Q) \leq R(x,y) * R(y,x) \rightarrow \Omega(x,y)$. From here we deduce

$$Ord_{\Omega}(Q) \leq \bigwedge_{x,y \in X} [R(x,y) * R(y,x) \rightarrow \Omega(x,y)] = Ant_{\Omega}(R).$$

We still have to prove

(c) $Ord_{\Omega}(Q) \leq Trans(R)$.

Let $x, y \in X$. Then

$$Ord_{\Omega}(Q) * R(x,y) * R(y,z) = \bigvee_{i,j \in I} Ord_{\Omega}(Q) * R_i(x,y) * R_j(y,z)$$

Let $i, j \in I$. Assume $R_i \preceq R_j$ therefore $R_i \subseteq R_j$ and $Ord_{\Omega}(R_i) \leq Ord_{\Omega}(R_j)$. According to Proposition 6 (2):

$$\begin{aligned} Ord_{\Omega}(Q) * R_i(x,y) * R_j(y,z) &\leq Ord_{\Omega}(R_j) * R_j(x,y) * R_j(y,z) \leq \\ &\leq Trans(R_j) * R_j(x,y) * R_j(y,z) \leq R_j(x,z) \leq R(x,z) \end{aligned}$$

from where, $Ord_{\Omega}(Q) * R(x,y) * R(y,z) \leq R(x,z)$. By applying Lemma 1 (1) it follows that for any $x, y, z \in X$ we have $Ord_{\Omega}(Q) \leq R(x,y) * R(y,z) \rightarrow R(x,z)$, from where

$$Ord_{\Omega}(Q) \leq \bigwedge_{x,y,z \in X} [R(x,y) * R(y,z) \rightarrow R(x,z)] = Trans(R).$$

From (a), (b) and (c) one obtains $Ord_{\Omega}(Q) \leq Ord_{\Omega}(R)$. We have shown that $(Ext(Q), \preceq)$ is inductive. According to Zorn's axiom a maximal element exists in $Ext(Q)$. \square

In the following we will situate ourselves in the case of the Gödel t-norm \wedge .

Theorem 8. *Let Q be a fuzzy relation on X . Then there exists a fuzzy relation R on X such that $Q \preceq R$ and $Ord_{\Omega}(Q) \leq SC(R)$.*

Proof. According to Lemma 7 there exists a fuzzy relation R on X maximal in $(Ext(Q), \preceq)$. Then $Q \preceq R$. It remains to prove that $Ord_{\Omega}(Q) \leq SC(R)$. We assume by absurdum that

$$Ord_{\Omega}(Q) \not\leq SC(R) = \bigwedge_{x,y \in X} (R(x,y) \vee R(y,x)).$$

therefore there exist $a, b \in X$ such that $Ord_{\Omega}(Q) \not\leq R(a,b) \vee R(b,a)$ from where $Ord_{\Omega}(Q) \not\leq R(a,b)$ and $Ord_{\Omega}(Q) \not\leq R(b,a)$. Assume $R(a,b) \leq R(b,a)$. According to the lines above, $R(b,a) < Ord_{\Omega}(Q)$.

We define a new fuzzy relation R' on X by

$$R'(x,y) = R(x,y) \vee (R(x,b) \wedge R(a,y))$$

for any $x, y \in X$. We intend to prove that $R \preceq R'$. It is obvious that $R \subseteq R'$ therefore it remains to prove that $Ord_{\Omega}(R) \leq Ord_{\Omega}(R')$. From $R \subseteq R'$ it follows immediately

(1) $Ord_{\Omega}(R) \leq Ref_{\Omega}(R')$

We establish now the inequality:

(2) $Ord_{\Omega}(R) \leq Ant_{\Omega}(R')$

Let $x, y \in X$. Then

$$\begin{aligned}
& Ord_{\Omega}(R) \wedge R'(x,y) \wedge R'(y,x) = \\
& = Ord_{\Omega}(R) \wedge [R(x,y) \vee (R(x,b) \wedge R(a,y))] \wedge [R(y,x) \vee (R(y,b) \wedge R(a,x))] = \\
& = [Ord_{\Omega}(R) \wedge R(x,y) \wedge R(y,x)] \vee [Ord_{\Omega}(R) \wedge R(x,y) \wedge R(y,b) \wedge R(a,x)] \vee \\
& \vee [Ord_{\Omega}(R) \wedge R(y,x) \wedge R(x,b) \wedge R(a,y)] \vee [Ord_{\Omega}(R) \wedge R(x,b) \wedge R(a,y) \wedge R(y,b) \wedge R(a,x)]
\end{aligned}$$

We will establish the following inequalities:

- (a) $Ord_{\Omega}(R) \wedge R(x,y) \wedge R(y,x) \leq \Omega(x,y)$;
- (b) $Ord_{\Omega}(R) \wedge R(x,y) \wedge R(y,b) \wedge R(a,x) \leq \Omega(x,y)$;
- (c) $Ord_{\Omega}(R) \wedge R(y,x) \wedge R(x,b) \wedge R(a,y) \leq \Omega(x,y)$;
- (d) $Ord_{\Omega}(R) \wedge R(x,b) \wedge R(a,y) \wedge R(y,b) \wedge R(a,x) \leq \Omega(x,y)$.

In order to obtain (a) we use Proposition 6 (4):

$$Ord_{\Omega}(R) \wedge R(x,y) \wedge R(y,x) \leq Ant_{\Omega}(R) \wedge R(x,y) \wedge R(y,x) \leq \Omega(a,b).$$

We treat now the other three cases. According to Proposition 6 (5) we have

$$Ord_{\Omega}(R) \leq \Omega(a,b) \leftrightarrow (R(a,b) \wedge R(b,a)) = \Omega(a,b) \leftrightarrow R(a,b)$$

We consider first the case $R(x,y) \leq R(y,x)$. Then $Ord_{\Omega}(R) \leq \Omega(x,y) \leftrightarrow R(x,y)$ with the same argument as above. (b) results like this:

$$Ord_{\Omega}(R) \wedge R(x,y) \wedge R(y,b) \wedge R(a,x) \leq R(x,y) \wedge [\Omega(x,y) \leftrightarrow R(x,y)] \leq \Omega(x,y)$$

Now we treat cases (c) and (d). First we notice that according to Proposition 6 (2):

$$Ord_{\Omega}(R) \wedge R(y,x) \wedge R(x,b) \wedge R(a,y) \leq Trans(R) \wedge R(a,y) \wedge R(y,x) \wedge R(x,b) \leq R(a,b)$$

therefore

$$Ord_{\Omega}(R) \wedge R(y,x) \wedge R(a,y) \leq R(a,b) \wedge [R(a,b) \leftrightarrow \Omega(a,b)] \leq \Omega(a,b).$$

Analogously we obtain:

$$Ord_{\Omega}(R) \wedge R(x,b) \wedge R(a,y) \wedge R(y,b) \wedge R(a,x) \leq \Omega(a,b).$$

We consider the possible subcases:

- (i) $\Omega(a,b) \leq R(x,y)$;
- (ii) $\Omega(a,b) > R(x,y)$.

According to the proof above, in case (i) the inequalities (c) and (d) are immediate. We are situated now in case (ii). One notices that

$$Ord_{\Omega}(R) \wedge R(x,b) \wedge \Omega(a,b) \wedge R(a,y) \leq \Omega(a,b) \wedge [\Omega(a,b) \leftrightarrow R(a,b)] \leq R(a,b) \leq R(b,a)$$

therefore

$$Ord_{\Omega}(R) \wedge R(x,b) \wedge \Omega(a,b) \wedge R(a,y) \leq Ord_{\Omega}(R) \wedge R(x,b) \wedge R(b,a) \wedge R(a,y) \leq Trans(R) \wedge R(x,b) \wedge R(b,a) \wedge R(a,y) \leq R(x,y)$$

From $Ord_{\Omega}(R) \wedge R(x,b) \wedge \Omega(a,b) \wedge R(a,y) \leq R(x,y)$ and $\Omega(a,b) > R(x,y)$ it follows

$$Ord_{\Omega}(R) \wedge R(x,b) \wedge R(a,y) \leq R(x,y).$$

By using this last inequality we have

$$Ord_{\Omega}(R) \wedge R(y,x) \wedge R(x,b) \wedge R(a,y) \leq Ord_{\Omega}(R) \wedge R(x,b) \wedge R(a,y) \leq R(x,y)$$

from where

$$Ord_{\Omega}(R) \wedge R(y,x) \wedge R(x,b) \wedge R(a,y) \leq R(x,y) \wedge [R(x,y) \leftrightarrow \Omega(x,y)] \leq \Omega(x,y)$$

Thus (c) was proved and (d) follows analogously.

The case $R(y,x) \leq R(x,y)$ is treated analogously. Therefore the inequalities (a)–(d) are true, so $Ord_{\Omega}(R) \wedge R'(x,y) \wedge R'(y,x) \leq \Omega(x,y)$. Cf. Lemma 1 (1) for any $x,y,z \in X$ we have $Ord_{\Omega}(R) \leq (R'(x,y) \wedge R'(y,x)) \rightarrow \Omega(x,y)$ therefore

$$Ord_{\Omega}(R) \leq \bigwedge_{x,y \in X} [(R'(x,y) \wedge R'(y,x)) \rightarrow \Omega(x,y)] = Ant_{\Omega}(R')$$

Now we establish the inequality

$$(3) \quad Ord_{\Omega}(R) \leq Trans(R').$$

Let $x,y,z \in X$. We prove

$$(4) \quad Ord_{\Omega}(R) \wedge R'(x,y) \wedge R'(y,z) \leq R'(x,z).$$

We notice that

$$Ord_{\Omega}(R) \wedge R'(x,y) \wedge R'(y,z) =$$

$$\begin{aligned}
 &= \text{Ord}_\Omega(R) \wedge [R(x, y) \vee (R(x, b) \wedge R(a, y))] \wedge [R(y, z) \vee (R(y, b) \wedge R(a, z))] = \\
 &= [\text{Ord}_\Omega(R) \wedge R(x, y) \wedge R(y, z)] \vee [\text{Ord}_\Omega(R) \wedge R(x, y) \wedge R(y, b) \wedge R(a, z)] \vee \\
 &\vee [\text{Ord}_\Omega(R) \wedge R(y, z) \wedge R(x, b) \wedge R(a, y)] \vee [\text{Ord}_\Omega(R) \wedge R(x, b) \wedge R(a, y) \wedge R(y, b) \wedge R(a, z)].
 \end{aligned}$$

Then to prove (4) is equivalent with establishing the following inequalities:

- (e) $\text{Ord}_\Omega(R) \wedge R(x, y) \wedge R(y, z) \leq R'(x, z)$;
- (f) $\text{Ord}_\Omega(R) \wedge R(x, y) \wedge R(y, b) \wedge R(a, z) \leq R'(x, z)$;
- (g) $\text{Ord}_\Omega(R) \wedge R(y, z) \wedge R(x, b) \wedge R(a, y) \leq R'(x, z)$;
- (h) $\text{Ord}_\Omega(R) \wedge R(x, b) \wedge R(a, y) \wedge R(y, b) \wedge R(a, z) \leq R'(x, z)$.

(e) follows by applying Proposition 6 (2):

$$\text{Ord}_\Omega(R) \wedge R(x, y) \wedge R(y, z) \leq \text{Trans}(R) \wedge R(x, y) \wedge R(y, z) \leq R(x, z) \leq R'(x, z).$$

(f) and (g) follow like this:

$$\text{Ord}_\Omega(R) \wedge R(x, y) \wedge R(y, b) \wedge R(a, z) \leq (\text{Trans}(R) \wedge R(x, y) \wedge R(y, b)) \wedge R(a, z) \leq R(x, b) \wedge R(a, z) \leq R'(x, z);$$

$$\text{Ord}_\Omega(R) \wedge R(y, z) \wedge R(x, b) \wedge R(a, y) \leq (\text{Trans}(R) \wedge R(a, y) \wedge R(y, z)) \wedge R(x, b) \leq R(x, b) \wedge R(a, z) \leq R'(x, z).$$

(h) follows similarly. We established (e)–(h), therefore (4) is true. Cf. Lemma 1 (1) for any $x, y, z \in X$ we have $\text{Ord}_\Omega(R) \leq (R'(x, y) \wedge R(y, z)) \rightarrow R'(x, z)$, from where

$$\text{Ord}_\Omega(R) \leq \bigwedge_{x, y, z \in X} [(R'(x, y) \wedge R'(y, z)) \rightarrow R'(x, z)] = \text{Trans}(R')$$

From (1), (2) and (3) we deduce $\text{Ord}_\Omega(R) \leq \text{Ord}_\Omega(R')$ therefore $R \preceq R'$. We can see that

$R'(a, b) = R(b, a) \vee (R(b, b) \wedge R(a, a))$ and $R(b, a) < \text{Ord}_\Omega(Q) \leq \text{Ord}_\Omega(R)$ (since $Q \preceq R$). Then, by applying Proposition 6 (6):

$$R(b, a) < \text{Ord}_\Omega(R) \leq R(b, b) \wedge R(a, a) \leq R'(a, b).$$

It follows $R \neq R'$, contradicting the maximality of R . We conclude $\text{Ord}_\Omega(Q) \leq \text{SC}(R)$, therefore the theorem is proved. \square

Remark 9. By Applying Lemma 5 we see that Theorem 3 is a particular case of Theorem 8.

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