

State Analysis of Time-Varying Singular Bilinear Systems by RK-Butcher Algorithms

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Abstract: The Runge-Kutta (RK)-Butcher algorithm is used to study the time-varying singular bilinear systems with the exact solutions. The results (discrete solutions) obtained using the Haar wavelets, Single-Term Walsh series (STWS) and RK-Butcher algorithms are compared with the exact solutions of the time-varying singular bilinear systems. It is found that the solution obtained using the RK-Butcher algorithm is closer to the exact solutions of the time-varying singular bilinear systems. The RK-Butcher algorithm can easily be implemented using a digital computer and the solution can be obtained for any length of time, which is an added advantage of this algorithm.

Keywords: Time-varying singular bilinear systems, Haar wavelets, Runge-Kutta Butcher algorithm, STWS algorithm.

1 Introduction

The development of singular bilinear systems has been studied by some researchers. Campbell [1] had done some preliminary work, but there was no available closed-form solution in that paper. In some analysis of neural networks, both singular systems [2] and bilinear systems [3] have been used. The multipliers and algebraic interconnections between singular systems and bilinear systems are allowed in dynamical systems. For singular bilinear systems, Lewis et al. [4] have been discussed extensively in the literature. However, the solution due to Lewis et al. only applies for the time-invariant case. Hsiao and Wang [5] applied the Haar wavelets for the solution of time-varying singular bilinear systems. Sepehrian and Razzaghi [6] applied the STWS approach for finding the numerical solution of time-varying bilinear systems.

Runge-Kutta (RK) methods have become very popular both as computational techniques and as a topic for research [7-12]. Butcher [8] derived the best RK pair, together with an error estimate, and in all statistical measures this approach is known as the RK-Butcher algorithm. This algorithm appears to be of sixth order because it requires six function evaluations, but in practice the 'working order' is closer to five (fifth order). However, the accuracy of the results obtained is better than that of all other algorithms examined including the RK-Fehlberg, RK-Merson, RK-centroidal mean (RKCeM) and RK-arithmetic mean (RKAM) algorithms.

Murugesh and Murugesan [13-15] introduced the RK-Butcher algorithm in Raster and Time-multiplexing CNN simulations. Recently, [16, 18] the RK-Butcher algorithm is used to find the numerical solution of an industrial application problem. In this article, we present a new approach for solving the time-varying singular bilinear systems using the RK-Butcher algorithm with more accuracy.

2 The RK-Butcher Algorithm

The normal order of an RK algorithm is the approximate number of leading terms of an infinite Taylor series which calculates the trajectory of a moving point [17]. The remainder of the infinite sum, which is excluded, is referred to as the local truncation error (LTE). These RK algorithms are forward-looking predictors, i.e. they do not use any information from preceding steps to predict the future position of a point. For this reason, they require a minimum of input data and consequently are very simple to program and use.

The general p -stage Runge-Kutta method for solving an IVP is

$$y' = f(x, y) \quad (1)$$

with the initial condition $y(x_0) = y_0$ is defined by

$$y_{n+1} = y_n + h \sum_{i=1}^p b_i k_i$$

where

$$k_i = f \left(x_n + c_i h, y_n + h \sum_{j=1}^p a_{ij} k_j \right), \quad i = 1, 2, 3, \dots, p$$

and

$$c_i = \sum_{j=1}^p a_{ij}, \quad i = 1, 2, 3, \dots, p$$

In the preceding equations c and b are p -dimensional vectors and $A(a_{ij})$ is the $p \times p$ matrix. Then the Butcher array takes the form

c_1	a_{11}			
c_2	a_{21}	a_{22}		
c_3	a_{31}	a_{32}	a_{33}	
\cdot	\cdot	\cdot	\cdot	
\cdot	\cdot	\cdot	\cdot	
\cdot	\cdot	\cdot	\cdot	\cdot
c_p	a_{p1}	a_{p2}	a_{p3}	a_{pp}
	b_1	b_2	b_{p-1}	b_p

The RK-Butcher algorithm of equation (1) is of the form

$$\begin{aligned}
 k_1 &= hf(x_n, y_n) \\
 k_2 &= hf\left(x_n + \frac{h}{4}, y_n + \frac{k_1}{4}\right) \\
 k_3 &= hf\left(x_n + \frac{h}{4}, y_n + \frac{k_1}{8} + \frac{k_2}{8}\right) \\
 k_4 &= hf\left(x_n + \frac{h}{2}, y_n - \frac{k_2}{2} + k_3\right) \\
 k_5 &= hf\left(x_n + \frac{3h}{4}, y_n + \frac{3k_1}{16} + \frac{9k_4}{16}\right) \\
 k_6 &= hf\left(x_n + h, y_n - \frac{3k_1}{7} + \frac{2k_2}{7} + \frac{12k_3}{7} - \frac{12k_4}{7} + \frac{8k_5}{7}\right)
 \end{aligned} \quad (2)$$

5th order predictor

$$y_{n+1} = y_n + \frac{1}{90} (7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6)$$

4th order predictor

$$y_{n+1}^* = y_n + \frac{1}{6} (k_1 + 4k_4 + k_6)$$

The local truncation error estimate (EE) is

$$EE = y_{n+1} - y_{n+1}^*$$

Then the formation of the Butcher array of the above equation (2) takes the following form

0						
$\frac{1}{4}$	$\frac{1}{4}$					
$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$				
$\frac{1}{2}$	0	$-\frac{1}{2}$	1			
$\frac{3}{4}$	$\frac{3}{16}$	0	0	$\frac{9}{16}$		
1	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{12}{7}$	$\frac{12}{7}$	$\frac{8}{7}$	
	$\frac{7}{90}$	0	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$
	$\frac{1}{6}$	0	0	$\frac{4}{6}$	0	$\frac{1}{6}$

3 Analysis of Time-Varying Singular Bilinear Systems

Consider the linear first order time-varying singular system

$$K\dot{x}(t) = Ax(t) + B(t)u(t) \tag{3}$$

with $x_0 = x(0)$, where K is an $n \times n$ singular matrix, A is an $n \times n$ matrix, B is an $n \times r$ matrix. $x(t)$ is n -state vectors and $u(t)$ is an r -input vector.

The time-varying singular bilinear system is of the form

$$E(t)\dot{x}(t) = A(t)x(t) + \sum_{i=1}^q N_i(t)x(t)u_i(t) + B(t)u(t) \tag{4}$$

equation (4) is written in the form (3) as

$$E(t)\dot{x}(t) = \left(A(t) + \sum_{i=1}^q N_i(t)u_i(t) \right) x(t) + B(t)u(t) \tag{5}$$

where the singular matrix $E(t) \in R^{n \times n}$, the state $x(t) \in R^n$, the control $u(t) \in R^q$, $A(t) \in R^{n \times n}$ and $B(t) \in R^{n \times q}$. $N_i(t) \in R^{n \times n}$ and $u_i(t), i = 1, 2, 3, \dots, q$, are the components of $u(t)$. The response $x(t), 0 \leq t \leq t_i$, is required to be found. The time-varying singular bilinear systems are much more difficult to solve than the time invariant singular bilinear systems. Therefore, many authors have tried various transform methods to overcome these difficulties. In this article, we introduce RK-Butcher algorithms with more accuracy to solve these time-varying singular bilinear systems.

4 Numerical Example

Consider the time-varying singular bilinear system of the following form (Hsiao and Wang [5]) and Seprehrian and Razzaghi[6]).

$$\begin{aligned}
 E(t) &= \begin{bmatrix} 0 & -t & 0 \\ 1 & 0 & t \\ 0 & 1 & 0 \end{bmatrix} & A(t) &= \begin{bmatrix} -2 & t & 1 \\ 0 & -4 & 2 \\ -2t & 0 & 1 \end{bmatrix} & N_1(t) &= \begin{bmatrix} 1 & -t & 1 \\ 0 & 3 & -2 \\ 2t & 0 & -2 \end{bmatrix} & (6) \\
 B(t) &= [2 \ 1 \ 3]^T, & u(t) &= 1, & \text{with initial condition } & x(0) = [12 \ 2 \ 5]^T
 \end{aligned}$$

when we solve (5), the analytic solution for $x(t)$ can be shown as

$$x(t) = \begin{bmatrix} (2-t) \left(\exp\left(\frac{-t}{2}\right) + \exp(t) \right) + 8 \\ 2\exp\left(\frac{-t}{2}\right) - \exp(t) + 1 \\ \exp\left(\frac{-t}{2}\right) + \exp(t) + 3 \end{bmatrix} \quad (7)$$

The discrete solutions of equation (5) are evaluated using the RK-Butcher algorithms (with step size $t = 0.25$) represented in equation (2) and the results are compared with the solutions obtained by the Haar wavelets method by Hsiao and Wang [5] and the STWS method by Sepehrian and Razzaghi [6]. The results are shown in tables 1 - 3 along with the exact solutions calculated using equation (7). Errors between the exact and discrete solutions are also given in Tables 1 - 3.

S.No	Time	Discrete solution x_1 Values						
		Exact Solutions	Haar Solutions	Haar Error	STWS Solutions	STWS Error	RK-Butcher Solutions	RK-Butcher Error
1	0	12.000000	12.0000	0.000000	12.00000	0.000000	12.000000	0.000000
2	0.25	11.886053	11.8861	0.000047	11.88605	0.000003	11.886053	0.000000
3	0.5	11.791414	11.7914	0.000014	11.79142	0.000006	11.791414	0.000000
4	0.75	11.711533	11.7115	0.000033	11.71154	0.000007	11.711533	0.000000
5	1	11.641283	11.6413	0.000017	11.64128	0.000003	11.641283	0.000000
6	1.25	11.574810	11.5748	0.000010	11.57481	0.000000	11.574810	0.000000
7	1.5	11.505362	11.5054	0.000038	11.50537	0.000008	11.505362	0.000000
8	1.75	11.425089	11.4251	0.000011	11.42510	0.000011	11.425089	0.000000
9	2	11.324812	11.3248	0.000012	11.32481	0.000002	11.324812	0.000000

Table 1: Solutions for the problem at various values of x_1 .

S.No	Time	Discrete solution x_2 Values						
		Exact Solutions	Haar Solutions	Haar Error	STWS Solutions	STWS Error	RK-Butcher Solutions	RK-Butcher Error
1	0	2.000000	2.0000	0.000000	2.00000	0.000000	2.000000	0.000000
2	0.25	1.745678	1.7457	0.000022	1.74568	0.000002	1.745678	0.000000
3	0.5	1.480968	1.4810	0.000032	1.48097	0.000002	1.480968	0.000000
4	0.75	1.203067	1.2031	0.000033	1.20307	0.000003	1.203067	0.000000
5	1	0.908880	0.9089	0.000020	0.90888	0.000000	0.908880	0.000000
6	1.25	0.594985	0.5950	0.000015	0.59498	0.000005	0.594985	0.000000
7	1.5	0.257579	0.2576	0.000021	0.25758	0.000001	0.257579	0.000000
8	1.75	-0.107578	-0.1076	0.000022	-0.10758	0.000002	-0.107578	0.000000
9	2	-0.505221	-0.5052	0.000021	-0.50522	0.000001	-0.505221	0.000000

Table 2: Solutions for the problem at various values of x_2 .

5 Conclusions

The discrete solutions obtained using the RK-Butcher algorithm gives more accurate values when compared to the Haar wavelets method discussed by Hsiao and Wang [5] and the STWS method by

S.No	Time	Discrete solution x_3 Values						
		Exact Solutions	Haar Solutions	Haar Error	STWS Solutions	STWS Error	RK-Butcher Solutions	RK-Butcher Error
1	0	5.000000	5.0000	0.000000	5.00000	0.000000	5.000000	0.000000
2	0.25	5.072562	5.0726	0.000038	5.07256	0.000002	5.072562	0.000000
3	0.5	5.166522	5.1665	0.000022	5.16652	0.000002	5.166522	0.000000
4	0.75	5.284021	5.2840	0.000021	5.28402	0.000001	5.284021	0.000000
5	1	5.427522	5.4275	0.000022	5.42752	0.000002	5.427522	0.000000
6	1.25	5.599862	5.5999	0.000038	5.59986	0.000002	5.599862	0.000000
7	1.5	5.804289	5.8043	0.000011	5.80429	0.000001	5.804289	0.000000
8	1.75	6.044524	6.0445	0.000024	6.04451	0.000014	6.044524	0.000000
9	2	6.324812	6.3248	0.000012	6.32480	0.000012	6.324812	0.000000

Table 3: Solutions for the problem at various values of x_3 .

Sepehrian and Razzaghi [6]. From the tables 1-3, one can observe that the solutions obtained by the RK-Butcher algorithm match well with the exact solutions of the time-varying singular bilinear systems, but the Haar wavelets and STWS methods yields a little error. Hence the RK-Butcher algorithm is more suitable for studying the time-varying bilinear systems.

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