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Grigore C. Moisil (1906 - 1973) and his School in Algebraic Logic

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Abstract: We present in the paper a very concise but updated survey emphasizing the research done by Gr. C. Moisil and his school in algebraic logic.
Keywords: n-valued Lukasiewicz-Moisil algebra, θ-valued Lukasiewicz-Moisil algebra, Post algebra

The mathematical logic is one of the domain in which the creative spirit of Gr.C. Moisil manifested plenary. His work in logic stands out by the the novelty, the variety and the depth of treated subjects. His first works are connected to the top results of the time and wear an algebraic seal. The young professor from Jassy came after a rich experience in mechanics and differential equations. Van der Waerden treatise of algebra has decisively influenced his entry in logic by the algebraic gate. In the same time, these works have a powerful philosophical imprint.

From this vast creation, the contributions in multiple-valued logics represent the part with the most intense impact on today researches.

The first system of multiple-valued logic was introduced by J. Łukasiewicz in 1920. Independently, E. Post introduced in 1921 a different multiple-valued logic. For Łukasiewicz, the motivation was of philosophical nature - he was looking for an interpretation of the concepts of possibility and necessity - while for Post, the research was intended as a natural mathematical generalization of bivalent logic.

In 1930, Łukasiewicz and Tarski studied a logic whose truth values are the real numbers from the interval $[0,1]$.

1 Łukasiewicz-Moisil algebras

In 1940, Gr. C. Moisil has defined the 3**-valued** and the 4**-valued Łukasiewicz algebras** and in 1942, the ***n*-valued Łukasiewicz algebras** ($n \geq 2$). His goal was to algebrize Łukasiewicz's logic. Boolean algebras, algebraic models of classical logic, are particular cases of that new structures.

In the description of a logical system, the implication was traditionally the principal connector. The n-valent system of Łukasiewicz had as truth values the elements of the set

$$L_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}, 1\right\}$$

and was built around a new concept of implication, on which are based the definitions of the other connectors.

For Moisil, the basic structure is that of lattice, to which he adds a negation (getting the so called "De Morgan algebra") and also some unary operations (called by Moisil "chryssipian endomorphisms"), representing the "nuancing". The Łukasiewicz implication was let on a secondary plane and, in the case of an arbitrary valence, was completely lost.

Further axiomatizations were suggested by A. Monteiro, R. Cignoli, C. Sicoe, S. Rudeanu and others.

An example of A. Rose from 1956 established that for $n \geq 5$ the Łukasiewicz implication can no more be defined on a Łukasiewicz algebra. Consequently, only for $n = 3$ and $n = 4$ the structures introduced by Moisil are models for Łukasiewicz logic. The lost of implication has lead to another type of logic, called today "Moisil logic", distinct from Łukasiewicz system; the logic corresponding to n-valued Łukasiewicz-Moisil algebras was created by Moisil in 1964. The fundamental concept of Moisil logic is the **nuancing**.

Nowadays we feel it appropiate to call these algebras **Łukasiewicz-Moisil algebras** or **LM algebras** for short.

For complete information and references on Łukasiewicz-Moisil algebras see [25]

The work of Moisil on LM algebras covers two periods of time: a first period, during 1940-1942, when he introduces the n-valued LM algebras with negation and studies special classes of these structures, as **centered** and **axed** LM algebras and a second one, during 1954-1973, when he introduces the θ-valued LM algebras without negation, applies multiple-valued logics to swiching theory and study algebraic properties of LM algebras (representation, ideals, reziduation).

Moisil's works traced research directions for many Romanian and foreign mathematicians. In Argentina, at Bahia Blanca, Antonio Monteiro and his school (Roberto Cignoli, Luiz Monteiro, Luiza Iturrioz, Maurice Abad etc.) have contributed decisively to consolidate LM agebras as a domain of algebra of logic and to disseminate them in the mathematical world.

In his PhD thesis from 1969 [29], R. Cignoli makes a very deep study of n-valued Moisil algebras (the name he first gives to the n-valued Łukasiewicz algebras introduced by Moisil).

1.1 n-valued Łukasiewicz-Moisil algebras

The structure called "De Morgan algebra" was first studied by Moisil; the name was given by Antonio Monteiro [142]; a duplicate name is "quasi-Boolean algebra" given by A. Bialynicki-Birula and H. Rasiowa.

Definition 1.1. *A* **De Morgan algebra** *is a structure*

$$(A, \vee, \wedge, ^-, 0, 1)$$

such that $(A, \vee, \wedge, 0, 1)$ *is a distributive lattice with 0 and 1 and the unary operation* $^-$*, called negation, verifies:*
(DM0) $1 = 0^-$,
(DM1) $(x^-)^- = x$,
(DM2) $(x \wedge y)^- = x^- \vee y^-$.

Remark 1.2. *In a De Morgan algebra we also have:*
(DM3) $(x \vee y)^- = x^- \wedge y^-$.

Definition 1.3. *Let* $J = \{1, 2, \ldots, n-1\}$.
*An n-***valued Łukasiewicz-Moisil algebra** $(n \geq 2)$ *or an* LM_n *bf algebra for short is an algebra*

$$\mathcal{A} = (A, \vee, \wedge, ^-, (r_j)_{j \in J}, 0, 1)$$

of type $(2, 2, 1, (1)_{j \in J}, 0, 0)$ *such that:*

(i) $(A, \vee, \wedge, ^-, 0, 1)$ *is a De Morgan algebra.*

(ii) the unary operations $r_1, r_2, \ldots, r_{n-1}$ *fulfil the following axioms: for every* $x, y \in A$ *and every* $i, j \in J$,

(L1) $r_j(x \vee y) = r_j x \vee r_j y$,

(L2) $r_j x \vee (r_j x)^- = 1$,

(L3) $r_j \circ r_i = r_i$,

(L4) $r_j(x^-) = (r_{n-j} x)^-$,

(L5) $r_1 x \leq r_2 x \leq \cdots \leq r_{n-1} x$,

(L6) if $r_j x = r_j y$ *for every* $j \in J$*, then* $x = y$*; this is* **the determination principle.**

If $\mathcal{A}$ *fulfils (i) and only (L1)–(L5) we shall say that* $\mathcal{A}$ *is an* LM_n **pre-algebra**.

Proposition 1.4. In every LM_n algebra $\mathcal{A}$, the following properties are verified: for every $x, y \in A$ and every $j \in J$,
(L7) $r_j(x \wedge y) = r_j x \wedge r_j y$;
(L8) $r_j x \wedge (r_j x)^- = 0$;
(L9) $x \leq y$ *if and only if* ($r_j x \leq r_j y$*, for every* $j \in J$);
(L10) $r_1 x \leq x \leq r_{n-1} x$;
(L11) $r_j 0 = 0$, $r_j 1 = 1$;
(L12) *Let* $C(A)$ *be the set of complemented elements of A, i.e.*

$$C(A) = \{x \in A \mid \exists x' \in A,\ x \vee x' = 1,\ x \wedge x' = 0\}.$$

Let K_j be the set of all elements of A left invariant by r_j, $j \in J$, i.e.

$$K_j = \{x \in A \mid r_j x = x\}.$$

Then:
(i) $r_j x \in C(A)$, for every $j \in J$, $x \in A$ and
(ii) $C(A) = K_j$, for every $j \in J$;
(L12') $(C(A), \vee, \wedge, ^-, 0, 1)$ is a Boolean algebra, where $x^- = x'$;
(L12") If $z \in C(A)$, then for every $x \in A$:
$x \wedge z = 0 \Longleftrightarrow x \leq z^-$,
$z \vee x = 1 \Longleftrightarrow z^- \leq x$;
(L13) $x^- \vee r_{n-1} x = 1$;
(L14) $x \wedge (r_{n-1} x)^- = 0$.

Example 1.5. The algebra

$$\mathcal{L}_n = \mathcal{L}_n^{(LM_n)} = (L_n, \vee, \wedge, ^-, (r_j)_{j \in J}, 0, 1),$$

where

$$L_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right\}$$

and

$$\begin{cases} x \vee y = \max(x,y), \quad x \wedge y = \min(x,y), \quad x^- = 1 - x, \\ r_j\left(\frac{i}{n-1}\right) = \begin{cases} 0, & \text{if } j+i < n, \\ 1, & \text{if } j+i \geq n, \end{cases} \quad i \in \{0\} \cup J,\ j \in J, \end{cases}$$

is an LM_n algebra, that we shall call **the canonical LM_n algebra**.

The proper subalgebras of $\mathcal{L}_n$ have the form:

$$S = L_n - \bigcup_{x \in L_n - \{0\}} \{x, x^-\}.$$

They are LM_n algebras.

The smallest subalgebra of $\mathcal{L}_n$ (with respect to $\subseteq$) is $C(L_n) = \{0, 1\}$, which is also a Boolean algebra, cf. (L12').

For instance, the subalgebras of
- L_3 are L_2 and L_3,
- L_4 are L_2 and L_4 and
- L_5 are L_2, L_3, $\{0, 1/4, 3/4, 1\}$ and L_5.

Remark 1.6. *LM_2 algebras coincide with Boolean algebras.*

Proposition 1.7. In every LM_n pre-algebra, the determination principle (L6) is equivalent to each of the following conditions: for every $x, y \in L$,

(a) $x \wedge (r_j x)^- \wedge r_{j+1} y \leq y$, *for every* $j \in J - \{n-1\}$;

(b) $x \wedge \bigwedge_{j=1}^{n-1} ((r_j x)^- \vee r_j y) \leq y$.

þ (Representation theorem of Moisil)
Every LM_n algebra can be embedded in a direct product of copies of the canonical LM_n algebra $\mathcal{L}_n$.

Corollary 1.1. Every LM_n algebra is a subdirect product of subalgebras of the canonical LM_n algebra $\mathcal{L}_n$.

* * *

In 1968, Gr. C.Moisil introduced the **θ-valued Łukasiewicz algebras or LM$_\theta$ algebras for short (without negation)**, where θ is the order type of a chain with first and last element. The concept of θ-valued Łukasiewicz algebra is obtained from that of n-valued, on the one hand, by dropping the negation $^-$ and on the other hand, by replacing the set L_n by a totally ordered set I with first and least elements and by adapting the axioms to this case; the Determination Principle is preserved. These structures were thought by Moisil as models of a logic with an infinity of nuances. According to a confession done by Moisil, he imagined LM$_\theta$ algebras (without negation) long time ago, but the care of finding a strong motivation for them delayed the announcement; the motivation was found when Moisil met Zadeh's fuzzy set theory, in which he saw a confirmation of his old ideas.

In 1969, Marek and Traczyk [110] introduced the notion of **generalized Łukasewicz algebra (with negation)**, in an attempt to generalize to the infinite case the LM_n algebras; but their generalization is not a natural one.

In his PhD thesis from 1972 [64], G. Georgescu studied duality theory for Moisil's LM_θ algebras (without negation), the injective objects (and their characterization), monadic and poliadic algebras.

In his PhD thesis from 1981 [53], A. Filipoiu studied the LM_θ algebras (without negation) and their associated logic. He gives a representation theorem for LM_θ algebras by aids of θ-valent Moisil field.

In his Master thesis from 1981 [12] also, L. Beznea studies a generalization of LM_θ algebras (without negation) obtained by eliminating the Determination Principle.

Later on, in his PhD theses from 1984 [21], V. Boicescu introduced and studied the ***n*-valued LM algebras without negation**, as a particular case of Moisil's LM_θ algebras (without negation).

Following the inverse way, A. Iorgulescu, in her PhD thesis from 1984 [90] also, introduced and studied a natural generalization of Moisil's LM_n algebras to the infinite case, called **θ-valued LM algebras with negation** or **LM_θ algebras with negation** for short; any LM_θ algebra with negation is a Moisil's LM_θ algebra without negation.

2 Connection with logic

Gr. C. Moisil invented LM algebras in order to create an algebraic structure playing the same role with respect to the multiple-valued logic as Boolean algebras play with respect to classical, bivalent logic. However, as shown by the example of A. Rose, this only happens for the cases $n = 3$ and $n = 4$.

The algebraic structures adequate to the infinite-valued logic of Łukasiewicz (truth valued in the real interval $[0,1]$) are the MV-algebras introduced by C.C. Chang in 1958 or, equivalently, the Wajsberg algebras introduced by Font, Rodriguez and Torrens in 1984; D. Mundici proved in in 1986 that MV algebras are categorically equivalent to lattice-ordered Abelian groups with strong unit.

R. Grigolia's MV_n algebras, introduced in 1977, and Cignoli's proper Łukasiewicz algebras, introduced in 1982, are algebraic structures corresponding to *n*-valued logic of Łukasiewicz.

The logic corresponding to LM_n algebras was created by Moisil himself in 1964. Łukasiewicz logic has implication as its primary connector, while Moisil logic is based on the idea of nuance, expressed algebraically by the Chrysippian endomorphisms. The “engine" of the latter logic is Moisil's Determination Principle, according to which an *n*-valued sentence is determined by its Boolean nuances. The Determination Principle realizes a transfer from the multiple-valued logic to the classical logic. This determination brings Moisil logic much closer to classical logic than Łukasiewicz logic. One could say that Moisil logic is **derived** from classical logic by the idea of nuancing. Algebraically, this tight relationship is expressed by the fundamental adjunction between the categories of Boolean and Łukasiewicz algebras.

V. Boicescu in 1971 and A. Filipoiu in 1981 introduced and studied logics appropiate to $LM\theta$ algebras without negation. (i.e. infinite-valued LM algebras).

A. Filipoiu generalized Smullyan's method of analytic tableaux to θ-valued logic without negation and studied the θ-valued predicate calculus as well, with applications to systems of recording and retrieval of information.

Łukasiewicz logic, Post logic and Moisil logic consitute the three directions in the classical theory of multiple-valued logic. Their corresponding algebraic models are MV algebras, Post algebras and LM algebras.

3 Connections with other structures of algebraic logic

Moisil introduced in 1941 the **centered** LM_3 algebras.

Post algebras (cf. P. Rosenbloom (1942), G. Epstein (1960), T. Traczyk (1963)) turn out to be centered LM algebras, cf. R. Cignoli (1969) and G. Georgescu - C. Vraciu (1969). The θ-valued Post algebras were studied by T. Traczyk (1967) and G. Georgescu (1971).

Gr. C. Moisil, R. Cignoli, L. Iturrioz, A. Monteiro and V. Boicescu studied LM algebras as particular cases of Heyting algebras. V. Boicescu also studied LM algebras as Stone algebras.

LM_3 algebras and LM_4 algebras are polynomially equivalent to MV_3 algebras and MV_4 algebras, respectively, since they are the algebraic counterpart of the 3-valued Łukasiewicz logic and the 4-valued Łukasiewicz logic, respectively. D. Mundici was first to point out the equivalence between LM_3 algebras and MV_3 algebras, in 1989. Then A. Iorgulescu, in 1998-2000 [91] - [94], pointed out the isomorphism between the categories of LM_k algebras and of MV_k algebras, for $k = 3, 4$ and also studied the categories $\mathbf{LM}_n$ and $\mathbf{MV}_n$ for $n \geq 5$, showing that every MV_n can be made into an LM_n algebra. She then studied those LM_n algebras that can be viewed as MV_n algebras:

3.1 Connections between LM_n algebras and MV_n algebras

MV algebras were introduced by C.C. Chang, in 1958 [26]. A simplified list of axioms of MV algebras was given by Mangani [109], as follows:

Definition 3.1. *An MV* **algebra** *is an algebra*

$$\mathcal{A} = (A, \oplus, ^-, 0)$$

of type $(2, 1, 0)$*, where the following axioms are verified: for every* $x, y, z \in A$*,*
(MV1) $(A, \oplus, 0)$ *is an Abelian monoid,*
(MV2) $x \oplus 0^- = 0^-$*,*
(MV3) $(x^-)^- = x$*,*
(MV4) $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$*,*
where $x \cdot y = (x^- \oplus y^-)^-$*.*

Definition 3.2. *For any* $m \in \mathbb{N}$*, we have:*

(i) $0x = 0$ *and* $(m+1)x = mx \oplus x$*,*

(ii) $x^0 = 1$ *and* $x^{m+1} = x^m \cdot x$*.*

The MV_n algebras were introduced by Revaz Grigolia in 1977 [87], as follows.

Definition 3.3. *An* MV_n **algebra** $(n \geq 2)$ *is an MV algebra* $\mathcal{A} = (A, \oplus, ^-, 0)$*, whose operations fulfil the additional axioms:*
(M1) $(n-1)x \oplus x = (n-1)x$*,*
(M1') $x^{n-1} \cdot x = x^{n-1}$
and, if $n \geq 4$*, the axioms:*
(M2) $[(jx) \cdot (x^- \oplus [(j-1)x]^-)]^{n-1} = 0$*,*
(M2') $(n-1)[x^j \oplus (x^- \cdot [x^{j-1}]^-)] = 1$*,*
where $1 < j < n-1$ *and* j *does not divide* $n-1$*.*

Corollary 3.1. MV_2 algebras coincide with Boolean algebras.

Example 3.4. The MV algebra $\mathcal{L}_n = \mathcal{L}_n^{(MV_n)} = (L_n, \oplus, ^-, 0)$, where

$$L_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right\}$$

and for any $x, y \in L_n$:

$$x \oplus y = \min(1, x+y), \quad x \cdot y = \max(0, x+y-1), \quad x^- = 1-x$$

and

$$x \vee y = \max(x, y), \quad x \wedge y = \min(x, y),$$

is an MV_n algebra. We shall call it **the canonical** MV_n **algebra**.
Note that $B(L_n) = \{0, 1\}$.
The subalgebras of $\mathcal{L}_n$ are of the form:

$$S_m = \left\{0, \frac{K}{n-1}, \ldots, \frac{(m-2)K}{n-1}, 1\right\},$$

where $K = \frac{n-1}{m-1}$, if $m-1$ divides $n-1$.

The subalgebras S_m of $\mathcal{L}_n$ are isomorphic to $L_m = \left\{0, \frac{1}{m-1}, \ldots, \frac{m-2}{m-1}, 1\right\}$, if $m-1$ divides $n-1$, and they are MV_n algebras.

Hence $\mathcal{L}_m = (L_m, \oplus, \cdot, ^-, 0, 1)$ $(m \leq n)$ is an MV_n algebra if and only if $m-1$ divides $n-1$.

For instance, the subalgebras of:
- L_3 are L_2 and L_3,
- L_4 are L_2 and L_4 and
- L_5 are L_2, L_3 and L_5.

þ

Every MV_n algebra is a subdirect product of subalgebras of the canonical MV_n algebra $\mathcal{L}_n$.

D. Mundici was the first to prove in 1989 [152] that MV_3 algebras coincide with LM_3 algebras.

A. Iorgulescu has proved in 1998-2000 [91] - [94] that:
1 - MV_4 algebras coincide with LM_4 algebras,
2 - the canonical MV_n algebra coincides with the canonical LM_n algebra ($n \geq 2$),
3 - for $n \geq 5$, any MV_n algebra is a LM_n algebra,
4 - those LM_n algebras which are MV_n algebras, for every $n \geq 5$, are exactly Cignoli's proper n-valued Łukasiewicz algebras.

Here are for short the results 1-3:

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To obtain the transformation of an MV_n algebra into an LM_n algebra, for any $n \geq 3$, Iorgulescu used Suchoń's transformation [174]:

Suchoń defines Moisil operators $(\sigma_j)_{j\in J}$ ($\sigma_j = r_{n-j}$) of the canonical LM_n algebra $(n \geq 3)$ starting from the Łukasiewiczian implication $\rightarrow$ and from the negation $^-$. He puts

$$B_3(x) = (x^-) \rightarrow x \text{ and } B_{j+1}(x) = (x^-) \rightarrow B_j(x),\ j \geq 3. \tag{1}$$

Then he defines:

$$\sigma_1 x = B_n(x) \tag{2}$$

$$\text{and for } 1 < j \leq [n/2], \qquad \sigma_j x = \begin{cases} \sigma_{n-1}(B_{l+1}(x)),\ lj \geq n-1 \\ \sigma_{lj}(B_{l+1}(x)),\ lj < n-1, \end{cases} \tag{3}$$

where $l = \max\{m \mid m(j-1) < n-1\}$,

$$\text{while } \sigma_{n-j}(x) = (\sigma_j(x^-))^-, \text{ for } \quad 1 \leq j \leq [n/2]. \tag{4}$$

Suchoń's Moisil operators verify: $\quad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{n-1}$.

Remark 3.5. *If we want to use Suchoń's construction, it is convenient to consider not the MV algebra* $(A, \oplus, ^-, 0)$, *but the Wajsberg algebra,* $(A, \rightarrow, ^-, 1)$, *introduced by J. M. Font, A. J. Rodriguez and A. Torrens in 1984; MV algebras and Wajsberg algebras are isomorphic structures:*

- *if* $\mathcal{A} = (A, \rightarrow, ^-, 1)$ *is a Wajsberg algebra and if we define*

$$\alpha(\mathcal{A}) = (A, \oplus, ^-, 0)$$

by

$$x \oplus y = x^- \rightarrow y, \quad 0 = 1^-, \tag{5}$$

then $\alpha(\mathcal{A})$ *is an MV algebra.*

- *Conversely, if* $\mathcal{A} = (A, \oplus, ^-, 0)$ *is an MV algebra and if we define*

$$\beta(\mathcal{A}) = (A, \rightarrow, ^-, 1)$$

by

$$x \rightarrow y = x^- \oplus y, \tag{6}$$
$$1 = 0^-,$$

then $\beta(\mathcal{A})$ *is a Wajsberg algebra.*

- *The maps* α, β *are mutually inverse.*

It follows immediately by (1) that

$$B_3(x) = x \oplus x = 2x \quad \text{and} \quad B_{j+1}(x) = x \oplus B_j(x) = jx,\ j \geq 3. \tag{7}$$

By using Suchoń's construction, Iorgulescu then gave the following

Definition 3.6. *Let $\mathcal{A} = (A, \oplus, ^-, 0)$ be an MV_n algebra ($n \geq 3$). Define*

$$\Phi^S(\mathcal{A}) = (A, \vee, \wedge, ^-, (r_j)_{j\in J}, 0, 1)$$

by

$$x \vee y = x \cdot y^- \oplus y, \qquad x \wedge y = (x^- \vee y^-)^-,$$

$$r_{n-1}x = (n-1)x, \tag{8}$$

$$r_{n-j}x = \begin{cases} r_1(lx), & lj \geq n-1 \\ r_{n-lj}(lx), & lj < n-1, \end{cases} \tag{9}$$

for $1 < j \leq [n/2], \qquad l = max\{m \mid m(j-1) < n-1\}$,

$$r_j x = (r_{n-j}(x^-))^-, \qquad 1 \leq j \leq [n/2]. \tag{10}$$

Proposition 3.7. If $\mathcal{L}_n$ is the canonical MV_n algebra ($n \geq 3$), then $\Phi^S(\mathcal{L}_n)$ is the canonical LM_n algebra.

þ If $\mathcal{A}$ is an MV_n algebra ($n \geq 3$), then $\Phi^S(\mathcal{A})$ is an LM_n algebra.

* * *

Proposition 3.8. 1) Given the canonical LM_n algebra ($n \geq 3$)

$$\mathcal{L}_n = (L_n, \vee, \wedge, ^-, (r_j)_{j\in J}, 0, 1),$$

$$\text{define} \quad \Psi(\mathcal{L}_n) = (L_n, \oplus^n, ^-, 0) \quad \text{by :}$$

if $n = 2k+1$,

$$\begin{aligned} x \oplus^{2k+1} y \;=\; & (x \vee r_{2k}y) \wedge (y \vee r_{2k}x) \\ \wedge\; & (x^* \vee r_{2k-1}y) \wedge (y^* \vee r_{2k-1}x) \\ & \vdots \\ \wedge\; & (x^{(k-1)*} \vee r_{k+1}y) \wedge (y^{(k-1)*} \vee r_{k+1}x), \end{aligned} \tag{11}$$

if $n = 2k$,

$$\begin{aligned} x \oplus^{2k} y \;=\; & (x \vee r_{2k-1}y) \wedge (y \vee r_{2k-1}x) \\ \wedge\; & (x^* \vee r_{2k-2}y) \wedge (y^* \vee r_{2k-2}x) \\ & \vdots \\ \wedge\; & (x^{(k-1)*} \vee r_k y) \wedge (y^{(k-1)*} \vee r_k x), \end{aligned} \tag{12}$$

where x^* is the **successor** of x and

$$x^{2*} = (x^*)^* \quad , \quad x^{m*} = \left(x^{(m-1)*}\right)^* \tag{13}$$

Then $\Psi(\mathcal{L}_n)$ is the canonical MV_n algebra.

2) The maps Φ^S, from Proposition 3.7, and Ψ are mutually inverse.

Since for $n = 3$, in the canonical LM_3 algebra $\mathcal{L}_3$, the operation $\oplus$ is:

$$x \oplus y = (x \vee r_2 y) \wedge (y \vee r_2 x)$$

and for $n = 4$, in the canonical LM_4 algebra $\mathcal{L}_4$, the operation $\oplus$ is defined by:

$$x \oplus y = (x \vee r_3 y) \wedge (y \vee r_3 x) \wedge (x^* \vee r_2 y) \wedge (y^* \vee r_2 x) =$$

$$= (x \vee r_3 y) \wedge (y \vee r_3 x) \wedge (x^- \vee y^- \vee r_2 x \vee r_2 y),$$

it follows that the transformation Ψ is not polynomial for $n \geq 5$).

Those LM_n algebras which are MV_n algebras (i.e. for which the transformation Ψ is defined), for every $n \geq 5$, are exactly Cignoli's proper n-valued Łukasiewicz algebras [34], but the proof is very technical [94].

4 Representation theorems

Numerous representation theorems have been given for LM algebras. The first is due to Moisil himself and is reminiscent of the representation theorem for Boolean algebras: every LM_n algebra can be embadded into a Cartesian power of L_n.

In a modern vision [25], every LM algebra is a subdirect product of subalgebras of the algebra $In(I, L_2)$ of incresing functions from I to L_2. In particular every LM_n algebra is a subdirect product of subalgebras of $\mathcal{L}_n$ (Cignoli). $\mathcal{L}$ is a direct product of subalgebras of $\mathcal{L}_n$ if and only if it is complete and atomic (Boicescu 1984).

The representation by continuous functions studied by Cignoli, Boicescu and Filipoiu, means that for every LM_θ algebra without negation $\mathcal{L}$ there is a unique Boolean space X such that $\mathcal{L}$ is isomorphic to the algebra of all continuous functions $f : X \to In(I, L_2)$, where $In(I, L_2)$ is endowed with the topology having as basis the principal ideals and the principal filters generated by the characteristic functions of the sets $\{k \mid k > \alpha\}$, $\alpha \in \theta$.

The representation of LM_θ algebras without or with negation by Moisil fields of sets is due to Filipoiu. The Stone duality was extended from Boolean algebras to LM_θ algebras without or with negation with the aid of a suitable concept called LM_θ-valued Stone space (Cignoli, Georgescu, Iorgulescu), while the Priestley duality is based on a suitable adaptation of the concept of Priestley space (Filipoiu).

The representation of LM algebras as algebras of fuzzy sets was studied by D. Ponasse, J.L. Coulon and J. Coulon, S. Ribeyre and S. Rudeanu.

The representation of LM_n algebras by LM_3 algebras is legitimated by the "good" properties of the latter and was studied by A. Monteiro, L. Monteiro, F. Coppola, V. Boicescu and A. Iorgulescu.

5 Categorial aspects

The Stone and Priestley dualities are in fact equivalences of categories. Other categorial properties of LM algebras were studied. Here are a few samples.

The association of L with the Boolean alegebra $C(L)$ of complemented elements of L is extended to a functor $C : \mathbf{LM}\theta \to \mathbf{B}$, while the association of a Boolean algebra B with the algebra $In(I, B)$ is extended to a functor $T : \mathbf{B} \to \mathbf{LM}\theta$. Then C and T are adjoint functors, C is faithful and T is fully faithful. This yields in particular the representation theorem of Moisil. The construction of the functors C and T was given by Moisil himself.

The injective and projective objects have also been studied, for instance, an LM_θ algebra is injective if and only if it is a complete Post algebra (whose center is a complete Boolean algebra), cf. L. Monteiro, R. Cignoli, G. Georgescu and C. Vraciu, V. Boicescu.

6 Ideals and congruences

The study of the appropiate ideal and congruence theory for LM algebras was undertaken by Gr. C. Moisil, A. Monteiro, R. Cignoli, C. Sicoe. V. Boicescu introduced the concepts of θ-ideal and θ-congruence, the prime spectre. For instance, in the case of LM_n algebras without negation, the congruence lattice of $\mathcal{L}$ is a Boolean algebra (a Stone algebra) if and only if $\mathcal{L}$ is finite ($C(L)$ is a complete Boolean algebra).

7 Monadic and polyadic algebras

L. Monteiro and G. Georgescu studied the generalization to LM algebras of the monadic and polyadic Boolean algebras introduced by P.R. Halmos. Sample results: the representation of monadic LM algebras by functional monadic LM algebras and the semantic completeness for polyadic LM algebras. A paper of G. Geogescu, A. Iorgulescu and I. Leuştean investigates monadic MV_n algebras and closed MV_n algebras.

8 Miscellanea

Various other topics have also been studied. Thus:

V. Boicescu proved that the lattice of equational subclass of LM_n is a finite Heyting algebra. The study of atomic algebras and the characterization of simple algebras as subalgebras of $In(I,L_2)$ and the property that LM_n algebras without negation form an equational class, are also due to Boicescu. The study of irredundant algebras and of exactly n-valued algebras is due to Boicescu as well.

A. Iorgulescu introduced and studied m-complete LM_θ algebras with negations, generalizing many of the properties of m-complete Boolean algebras.

G. Georgescu and I. Leuştean studied probabilities on LM algebras.

L. Beznea studied a generalization of LM algebras, obtained by dropping the determination principle.

Let us also mention M. Sularia's theory of D algebras. These structures are subdirect products between a Heyting algebra and a Brouwer algebra and represent the algebraic counterpart of a logic of problem solving.

In [44], [48], the authors study Łukasiewicz BCK algebras endowed with Moisil oparators.

In [163], [164], C. Sanza introduced and studied (monadic) $n \times m$-valued Łukasiewicz algebras with negation.

In a very recent paper [103], I. Leuştean proposes a unifying framework for LM_n algebras, MV algebras and Post algebras; essentially, an LM_{n+1} algebra is charcterized by a string of n Boolean ideals of his Boolean center. The necessary and sufficient conditions are given that such a string must satisfy to define a MV_{n+1} algebra or a Post algebra of order $n+1$. This result could be seen as a generalization of Moisil's Determination Principle. As an application, in paper [75], some special Cauchy completions of MV_{n+1} algebras are characterized by using the properties of corresponding strings of Boolean ideals.

In another very recent paper [75], G.Georgescu and A. Popescu introduced the notion of ***n*-nuanced MV algebra**, by performing a Łukasiewicz-Moisil nuancing construction on top of MV-algebras. These structures extend both MV-algebras and Łukasiewicz-Moisil algebras, thus unifying two important types of structures in the algebra of logic. On a logical level, n-nuanced MV algebras amalgamate two distinct approaches to many-valuedness: that of the infinitely valued Łukasiewicz logic, more related in spirit to the fuzzy approach, and that of Moisil n-nuanced logic, which is more concerned with **nuances of truth** rather than **truth degrees**. They study n-nuanced MV algebras mainly from the algebraic and categorial points of view and also consider some basic model-theoretic aspects. The relationship with a suitable notion of n-nuanced ordered group via an extension of the Γ construction is also analyzed:

8.1 *n*-nuanced MV algebras

Usually, MV algebras are defined only in terms of $\oplus$, $^-$ and 0. However, in order to point out the symmetry of these structures, the authors prefered the following slightly redundant definition:

Definition 8.1. *An* **MV algebra** *is a structure* $(A,\oplus,\odot,^-,0,1)$*, satisfying the following axioms:*

(MV1') $(A,\oplus,0)$ *and* $(A,\odot,1)$ *are commutative monoids,*

(MV2') $x\odot 0=0$ *and* $x\oplus 1=1$*,*

(MV3') $(x^-)^-=x$*,*

(MV4') $(x\oplus y)^-=x^-\odot y^-$*,*

(MV5') $(x\odot y^-)\oplus y=(y\odot x^-)\oplus x$*.*

Definition 8.2. *A* **generalized De Morgan algebra** *is a structure* $\mathcal{L}=(L,\oplus,\odot,^-,0,1)$*, where* $\oplus$*,* $\odot$ *are binary operations,* $^-$ *is a unary operation, and* $0,1$ *are constants such that the following conditions hold:*

(i) $(L,\oplus,0)$, $(L,\odot,1)$ *are commutative monoids;*

(ii) $(x\oplus y)^-=x^-\odot y^-$ *and* $(x^-)^-=x$ *for all* $x,y\in L$*; .*

Remark 8.3. *If* $\mathcal{L}$ *is a generalized De Morgan algebra, then* $(x\odot y)^-=x^-\oplus y^-$ *for all* $x,y\in L$*.*

Definition 8.4. *An* ***n*-nuanced MV-algebra (*NMV_n* algebra** *for short) is a structure*

$$\mathcal{L}=(L,\oplus,\odot,^-,r_1,\ldots,r_{n-1},0,1)$$

such that $(L,\oplus,\odot,^-,0,1)$ *is a generalized De Morgan algebra and* $r_1,\ldots,r_{n-1}$ *satisfy the following axioms:*

(A0) $r_ix\oplus((r_ix)^-\odot r_iy)=r_iy\oplus((r_iy)^-\odot r_ix)$*, for* $i\in\{1,\ldots,n-1\}$*,*

(A1) $r_i(x \oplus y) = r_i x \oplus r_i y$, $r_i(x \odot y) = r_i x \odot r_i y$, $r_i(0) = 0$, $r_i(1) = 1$, *for* $i \in \{1, \ldots, n-1\}$,
(A2) $r_i x \oplus (r_i x)^- = 1$, $r_i x \odot (r_i x)^- = 0$, *for* $i \in \{1, \ldots, n-1\}$,
(A3) $r_i \circ r_j = r_j$, *for* $i, j \in \{1, \ldots, n-1\}$,
(A4) $r_i(x^-) = (r_{n-i} x)^-$, *for* $i \in \{1, \ldots, n-1\}$,
(A5) (Determination Principle:) if $r_i x = r_i y$ *for each* $i \in \{1, \ldots, n-1\}$, *then* $x = y$,
(A6) $r_1 x \leq r_2 x \leq \ldots \leq r_{n-1} x$.

Remark 8.5. *NMV_n algebras provide a common generalization of MV- and Łukasiwicz-Moisil algebras. Indeed,*
- if $n = 2$, then, because of the Determination Principle, r_1 is the identity, thus an NMV_n algebra can be identified with an MV-algebra;
- if $(L, \oplus, \odot, ^-, 0, 1)$ is a De Morgan algebra, then $(L, \oplus, \odot, ^-, r_1, \ldots, r_{n-1}, 0, 1)$ becomes an LM_n algebra.

Example 8.6. *Let $\mathcal{A} = (A, \oplus, \odot, ^-, 0, 1)$ be an MV-algebra. Consider the set*

$$T(A) = \{(x_1, \ldots, x_{n-1}) \in A^{n-1} \mid x_1 \leq \ldots \leq x_{n-1}\}.$$

Since A^{n-1} is an MV-algebra (with operations taken component-wise from A) and $T(A)$ is closed under the operations $\mathbf{0}$, $\mathbf{1}$, $\oplus$, $\odot$ (where $\mathbf{0}$ and $\mathbf{1}$ are the constant vectors), then we can consider these operations on $T(A)$. We furthermore define $^-$, $r_1, \ldots, r_{n-1}$ by:
$(x_1, \ldots, x_{n-1})^- = (x_{n-1}^-, \ldots, x_1^-)$,
$r_i(x_1, \ldots, x_{n-1}) = (x_i, \ldots, x_i)$, *for* $i \in \{1, \ldots, n-1\}$.
Then $(T(A), \oplus, \odot, ^-, r_1, \ldots, r_{n-1}, \mathbf{0}, \mathbf{1})$ is an NMV_n algebra.

Define

$$M(L) = \{x \in L \mid r_i x = x \text{ for all } i \in \{1, \ldots, n-1\}\}.$$

Then $M(L)$, together with the operations $\oplus, \odot, ^-, 0, 1$ induced by $\mathcal{L}$, is an MV algebra, called **the MV-center** of $\mathcal{L}$.

In the MV-algebra $M(L)$ we have a canonical order $\leq$. Let us define an extension of this order to L by:

$x \leq y$ iff, for each $i \in \{1, \ldots, n-1\}$, $r_i x \leq r_i y$.

Because of the Determination Principle, this is indeed an order and because of (A3), it is indeed an extension of the order on $M(L)$. Moreover, the compatibility properties listed in the following lemma are obvious:

Proposition 8.7. *The following properties are true in a $\mathcal{L}$:*
(1) 0 *is the greatest and* 1 *is the least element in* L *w.r.t.* $\leq$;
(2) for each $x, y \in L$, $x \leq y$ *iff* $y^- \leq x^-$;
(3) for each $x, x', y, y' \in L$, *if* $x \leq x'$ *and* $y \leq y'$, *then* $x \oplus y \leq x' \oplus y'$ *and* $x \odot y \leq x' \odot y'$,
(4) $r_1 x \leq x \leq r_{n-1} x$, *for any* $x \in L$,
(5) for $x, y \in L$, *if* $x \oplus y = 1$ *and* $x \odot y = 0$, *then* $x, y \in M(L)$ *and* $y = x^-$,
(6) $M(L) = \{x \in L \mid x \oplus x^- = 1', \quad x \odot x^- = 0\}$.

9 Applications to switching theory

Whereas Boolean algebra is a suitable tool for the study of networks made up of binary devices, the study of networks involving multi-positional devices and the so-called hazard and race phenomena have imposed the use of other algebraic tools, namely Galois fields, Łukasiewicz-Moisil algebras and the theory of discrete functions.

Moisil investigated circuits involving devices such as polarized relays with unstable neutral, ordinary relays under low self-maintaining current, valves, resistances, multi-positional relays, as well as transistors and other electronic devices. See also [125]. Moisil has described the operation of such devices by characteristic equations of the form $x_{k+1} = \varphi(\xi_k, x_k)$, where the variable x associated with the relay contact takes values in L_n, where $n \leq 5$

depends on the type of the relay, $\xi \in L_2$ is a variable associated with the current and the index k or $k+1$ indicates the value of the corresponing variable at time $t = k$ or $t = k+1$, respectively.

The **synthesis problem** consists in designing a circuit made up of several relays and whose operation be described by a given equation of the form

$$X_{k+1} = F(A_k, X_k), \tag{14}$$

where X is the vector of the variables x associated with the relays of the circuit, A is the input vector and the meaning of the index k or $k+1$ is the same as above. To solve this problem, Moisil notices the crucial point that the structure of such a circuit is determined by the expression of a function G which satisfies the identity

$$\Xi = G(A, X), \tag{15}$$

where Ξ is the vector of the variables ξ associated with the relay of the circuit. So if

$$X_{k+1} = Phi(\Xi_k, X_k) \tag{16}$$

is the vector form of the characteristic equations of the relays of the circuit, it follows from (14) and (15) that

$$F(A_k, X_k) = \Phi(\Xi_k, X_k), \tag{17}$$

for any k. Therefore (15) transforms (17) into the identity

$$F(A, X) = \Phi(G(A, X), X), \tag{18}$$

and Moisil's method for solving the synthesis problem consists in solving the functional equation (18) with respect to G.

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