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# On Polar, Trivially Perfect Graphs

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**Abstract:** During the last decades, different types of decompositions have been processed in the field of graph theory. In various problems, for example in the construction of recognition algorithms, frequently appears the so-called weakly decomposition of graphs.
Polar graphs are a natural extension of some classes of graphs like bipartite graphs, split graphs and complements of bipartite graphs. Recognizing a polar graph is known to be NP-complete. For this class of graphs, polynomial algorithms for the maximum stable set problem are unknown and algorithms for the dominating set problem are also NP-complete.
In this paper we characterize the polar graphs using the weakly decomposition, give a polynomial time algorithm for recognizing graphs that are both trivially perfect and polar, and directly calculate the domination number. For the stability number and clique number, we give polynomial time algorithms.
**Keywords:** Polar graphs, trivial perfect graphs, weakly decomposition, recognition algorithms, optimization algorithms.

## 1 Introduction

Polar graphs are a natural extension of some classes of graphs like bipartite graphs, split graphs and complements of bipartite graphs.

According to ( [3]), a graph $G = (V, E)$ is called $\mathtt{polar}$ if the set $V$ of its vertices can be partitioned into $(S,Q)$ ($S$ or $Q$ possibly empty) such that $S$ induces a complete multipartite graph (that is a join of stable sets) and $Q$ is a disjoint union of cliques. In ( [3]) has been proved that the problem of recognizing an arbitrary graph to be polar is NP-complete.

Recently some important result concerning polar graphs have been proven. Hereby, in ( [9]) the authors give a characterization through forbidden subgraphs of polar cographs and a polynomial algorithm that finds the largest induced subgraph in a cograph. In ( [8]) is presented a polynomial algorithm to recognize the polar property for triangulated graphs. In ( [7]), a polynomial algorithm for the recognition of graphs that are both polar and permutation is given. In ( [16]) they assert that polynomial algorithms for independent set are unknown and algorithms for domination number are NP-complete for split graphs (see ( [2] and [4]).

Both problems to find independent maximal set of maximum and minimum weight are NP-hard, in general. In ( [12]) are given polynomial time algorithms that solve the problems formulated above for classes of polar graphs.

## 2 Definition and notation

Throughout this paper, $G = (V, E)$ is a connected, finite and undirected graph, without loops and multiple edges ( [1]), having $V = V(G)$ as the vertex set and $E = E(G)$ as the set of edges. $\overline{G}$ is the complement of $G$. If $U \subseteq V$, by $G(U)$ or $[U]_G$ we denote the subgraph of $G$ induced by $U$. By $G - X$ we mean the subgraph $G(V - X)$, whenever $X \subseteq V$, but we simply write $G - v$, when

$X = \{v\}$. If $e = xy$ is an edge of a graph $G$, then $x$ and $y$ are adjacent, while $x$ and $e$ are incident, as are $y$ and $e$. If $xy \in E$, we also use $x \sim y$, and $x \not\sim y$ whenever $x, y$ are not adjacent in $G$. A vertex $z \in V$ distinguishes the non-adjacent vertices $x, y \in V$ if $zx \in E$ and $zy \notin E$. If $A, B \subset V$ are disjoint and $ab \in E$ for every $a \in A$ and $b \in B$, we say that $A, B$ are *totally adjacent* and we denote by $A \sim B$, while by $A \not\sim B$ we mean that no edge of $G$ joins some vertex of $A$ to a vertex from $B$ and, in this case, we say that $A$ and $B$ are *non-adjacent*.

The *neighbourhood* of the vertex $v \in V$ is the set $N_G(v) = \{u \in V : uv \in E\}$, while $N_G[v] = N_G(v) \cup \{v\}$; we simply write $N(v)$ and $N[v]$, when $G$ appears clearly from the context. The neighbourhood of the vertex $v$ in the complement of $G$ will be denoted by $\overline{N}(v)$.

If $D \subset V$ and every vertex from $V - D$ has at least one neighbour in $D$, then $D$ is called a *dominating set* of $G$. The minimum size of a dominating set is the $\mathtt{domination\ number}$ $\nu(G)$.

A $\mathtt{complete}$ graph is a graph in which every vertex is adjacent to every other.

The neighbourhood of $S \subset V$ is the set $N(S) = \cup_{v \in S} N(v) - S$ and $N[S] = S \cup N(S)$. A *clique* is a subset $Q$ of $V$ with the property that $G(Q)$ is complete. The *clique number* of $G$, denoted by $\omega(G)$, is the size of the maximum clique.

An $\mathtt{independent}$ set or $\mathtt{stable}$ set is a set of vertices of which no pair is adjacent. The $\mathtt{independence\ number}$ $\alpha(G)$ of a graph $G$ is the size of a largest independent set of $G$.

By $P_n$, $C_n$, $K_n$ we mean a chordless path on $n \geq 3$ vertices, a chordless cycle on $n \geq 3$ vertices, and a complete graph on $n \geq 1$ vertices, respectively.

The $\mathtt{distance}$ $d_G(u, v)$ between two (not necessary distinct) vertices $u$ and $v$ in a graph $G$ is the length of a shortest path between them.

A graph is called $\mathtt{triangulated}$ if it does not contain chordless cycles having the length greater or equal to four.

A graph is called $\mathtt{cograph}$ if it does not contain $P_4$.

A graph is a $\mathtt{split}$ graph if the vertex set can be partitioned into a clique and a stable set.

A graph $G$ is $\mathtt{trivially\ perfect}$ ( [10]) if for each induced subgraph $H$ of $G$, the number of maximal cliques of $H$ is equal to the maximum size of an independent set of $H$.

Let $n \geq 1$ and $\pi$ be a permutation over $\{1, ..., n\}$. We will denote $\pi$ equivalently as a $\mathtt{permutation}$ sequence $(\pi(1), ..., \pi(n))$. The $\mathtt{inversion}$ graph of $\pi$ has vertex set $\{1, ..., n\}$ and two vertices $u$, $v$ are adjacent if $(u - v)(\pi^{-1}(u) - \pi^{-1}(v)) < 0$. A graph is a $\mathtt{permutation}$ graph if it is isomorphic to the inversion graph of a permutation sequence.

Let $F$ denote a family of graphs. A graph $G$ is called $F$-*free* if none of its subgraphs is in $F$. The *Zykov sum* of the graphs $G_1, G_2$ is the graph $G = G_1 + G_2$ having:

$$V(G) = V(G_1) \cup V(G_2),$$
$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$

When searching for recognition algorithms, frequently appears a type of partition for the set of vertices in three classes $A, B, C$, which we call a *weakly decomposition*, such that: $A$ induces a connected subgraph, $C$ is totally adjacent to $B$, while $C$ and $A$ are totally nonadjacent.

The structure of the paper is the following. In Section 3 we recall the notion of weakly decomposition. In Section 4 we present a new characterization of polar, trivially perfect graphs. In Section 5 we give a recognition algorithm for this class of graphs. In Section 6 we give combinatorial optimization algorithms for polar, trivially perfect graphs. In the last section we have our concluding remarks.

## 3 Preliminary results

At first, we recall the notions of weakly component and weakly decomposition.

**Definition 1.** ( [6], [13], [14]) *A set* $A \subset V(G)$ *is called a weakly set of the graph* $G$ *if* $N_G(A) \neq V(G) - A$ *and* $G(A)$ *is connected. If* $A$ *is a weakly set, maximal with respect to set inclusion, then* $G(A)$ *is called a weakly component. For simplicity, the weakly component* $G(A)$ *will be denoted with* $A$.

**Definition 2.** ( [6], [13], [14]) *Let* $G = (V, E)$ *be a connected and non-complete graph. If* $A$ *is a weakly set, then the partition* $\{A, N(A), V - A \cup N(A)\}$ *is called a weakly decomposition of* $G$ *with respect to* $A$.

Below we remind a characterization of the weakly decomposition of a graph.

The name of "*weakly component*" is justified by the following result.

**Theorem 1.** ( [5], [13], [14]) *Every connected and non-complete graph* $G = (V, E)$ *admits a weakly component* $A$ *such that* $G(V - A) = G(N(A)) + G(\overline{N}(A))$.

**Theorem 2.** ( [13], [14]) *Let* $G = (V, E)$ *be a connected and non-complete graph and* $A \subset V$. *Then* $A$ *is a weakly component of* $G$ *if and only if* $G(A)$ *is connected and* $N(A) \sim \overline{N}(A)$.

The next result, that follows from Theorem 1, ensures the existence of a weakly decomposition in a connected and non-complete graph.

**Corollary 1.** *If* $G = (V, E)$ *is a connected and non-complete graph, then* $V$ *admits a weakly decomposition* $(A, B, C)$, *such that* $G(A)$ *is a weakly component and* $G(V - A) = G(B) + G(C)$.

Theorem 2 provides an $O(n + m)$ algorithm for building a weakly decomposition for a non-complete and connected graph.

**Algorithm for the weakly decomposition of a graph** ( [13])
*Input:* A connected graph with at least two nonadjacent vertices, $G = (V, E)$.
*Output:* A partition $V = (A, N, R)$ such that $G(A)$ is connected, $N = N(A)$, $A \not\sim R = \overline{N}(A)$.

```
begin
   A := any set of vertices such that
   A ∪ N(A) ≠ V
   N := N(A)
   R := V − A ∪ N(A)
   while (∃n ∈ N, ∃r ∈ R such that nr ∉ E ) do
      begin
         A := A ∪ {n}
         N := (N − {n}) ∪ (N(n) ∩ R)
         R := R − (N(n) ∩ R)
      end
end
```

**Corollary 2.** *For* $G = (V, E)$ *a connected non-complete graph, and* $(A, N, R)$ *a weakly decomposition with* $G(A)$ *the weakly component the following relation holds:*

$\alpha(G) = max\{\alpha(G(A)) + \alpha(G(R)), \alpha(G(A \cup N))\}$ .
In ( [13]) some applications of weakly decomposition have been depicted. Let $G = (V, E)$ be connected, non-complete graph and $(A, N, R)$ a weakly decomposition, with $A$ the weakly component.The following hold:
a) $G$ is $P_4 - free$ if and only if $A \sim N \sim R$ and $G(A)$, $G(N)$ and $G(R)$ are $P_4 - free$;
b) $G$ is triangulated if and only if $N$ is a clique and $R$ and $G - R$ are triangulated.
Each of the results above lead to recognition algorithms for the specified graphs.
c) If $G$ is triangulated then $\alpha(G) = \alpha(G(A)) + \alpha(G(R))$ and this leads to the algorithm that determines $\alpha(G)$.

# 4 Characterization of polar, trivially perfect graphs

In this section, using the weakly decomposition, we present a recognition algorithm for the polar trivially perfect graphs. At first we remind a characterization in terms of forbidden subgraphs of polar cographs and two characterization of trivially perfect graphs.

**Theorem 3.** ( [9]) *For a cograph* $G$*, the following statements are equivalent:*
*a)* $G$ *is polar;*
*b) Neither* $G$ *nor* $\overline{G}$ *contains any one of the graphs* $H_1$*,* $H_2$*,* $H_3$*,* $H_4$ *as induced subgraphs, where* $H_i = G_i \cup F_i$ *(*$1 \leq i \leq 4$*), and every* $G_i$ *is a* $P_3$ *and* $F_i$*, described as sequences of degrees, are:* $F_1 : (4,3,3,3,3)$*;* $F_2 : (5,3,2,2,2,2)$*;* $F_3 : (4,4,3,3,3,3)$*;* $F_4 : (5,5,3,3,3,3)$.

**Theorem 4.** ( [15]) *If* $G$ *is a connected, non-complete graph and* $(A, N, R)$ *is a weakly decomposition with* $G(A)$ *the weakly component, then* $G$ *is trivially perfect if and only if:*
*i)* $A \sim N \sim R$*;*
*ii)* $N$ *is clique;*
*iii)* $G(A)$*,* $G(R)$ *are trivially perfect if and only if it contains no vertex subset that induces* $P_4$ *or* $C_4$.

**Theorem 5.** ( [12]) *A graph is trivially perfect if and only if it contains no vertex subset that induces* $P_4$ *or* $C_4$*.*

**Theorem 6.** *Let* $G = (V, E)$ *be a connected, non-complete graph and* $(A, N, R)$ *a weakly decomposition with* $G(A)$ *the weakly component. Let also* $G$ *and* $\overline{G}$ *be trivially perfect graphs.* $G$ *is polar if and only if* $G(A)$ *and* $G(R)$ *are polar graphs.*

*Proof.* If $G$ is a polar graph then $G(A)$ and $G(R)$ are polar graphs, as every induced subgraphs of a polar graph is also polar. Conversely, suppose that $G(A)$ and $G(R)$ are polar graphs. We show that $G$ is a polar graph. Because $G$ is trivially perfect it follows that $A \sim N \sim R$ and $N$ is a clique. Suppose that $X \subset V$ still exists such that $G(X)$ is isomorphic to one of the following four graphs: $H_1$, $H_2$, $H_3$, $H_4$, where $H_i = G_i \cup F_i$ $(1 \leq i \leq 4)$, and every $G_i$ is a $P_3$ and every $F_i$, described as a sequence of degrees, is: $F_1 : (4,3,3,3,3)$; $F_2 : (5,3,2,2,2,2)$; $F_3 : (4,4,3,3,3,3)$; $F_4 : (5,5,3,3,3,3)$. Because $G$ is trivially perfect it follows that $G$ is $\{P_4, C_4\}$-free. If $x$ is the vertex of degree 4 in $F_1$, $z$ and $t$ are the vertices of degree 4 in $F_3$, $u$ and $v$ are the vertices of degree 5 in $F_4$ then $F_1 - \{x\}$ is isomorphic to $C_4$, $F_3 - \{z, t\}$ is isomorphic to $C_4$, $F_4 - \{u, v\}$ is isomorphic to $C_4$, which is a contradiction. We know that the complement of a polar graph is a polar graph and that $\overline{G}$ is $C_4$-free, because it is trivially perfect. If $y$ is the vertex of degree 5 in $F_2$ and $a$ is one of the vertices of degree 2 adjacent to the vertex of degree 3 in $F_2$ then $F_2 - \{a, y\}$ is isomorphic to $C_4$, which is a contradiction.

# 5 Recognition of polar trivially perfect graphs

Theorem 6 leads to the following recognition algorithm.

*Input:* $G = (V, E)$ a connected graph satisfying the conditions in Theorem 6
*Output:* An answer to the question: Is $G$ a $\mathsf{Polar}$ graph ?

```
begin
    L = {G}  // L is a list of graphs
    while (L ≠ ∅)
        begin
          extract an element H from L
          find a weakly decomposition (A, N, R) for H
          if (A ≁ N ≁ R) then G is not trivially perfect
                              else  introduce in L the connected, non-complete components of
                                    G(A), G(R)
```

*end*
*Return:* $\mathsf{G}$ is $\mathsf{Polar}$
*end*

In what follows, we give some remarks on the algorithm.
Because the operation inside the body of $\mathtt{while}$ loop that takes the longest execution time is the weakly decomposition (namely $O(n+m)$) it follows that the total execution time of the algorithm is $O(n(n+m))$.

## 6 Combinatorial optimization algorithms for polar, trivially perfect graphs

In this section we calculate the domination number, give $O(n(n+m))$ algorithms to calculate the stability number and clique number.

Theorem 6 leads to the following result.

**Corollary 3.** *Let* $G = (V, E)$ *be a connected, non-complete graph and* $(A, N, R)$ *a weakly decomposition with* $G(A)$ *the weakly component. If* $G$ *is trivially perfect and also polar then the following hold:*
*i)* $\alpha(G) = \alpha(G(A)) + \alpha(G(R))$*;*
*ii)* $\omega(G) = |N| + max\{\omega(G(A)), \omega(G(R))\}$*;*
*iii)* $\nu(G) = 1$.

*Proof.* Let $T \subset A \cup N$ such that $T$ is stable and $|T| = \alpha(G(A \cup N))$. Because $N$ is a clique it follows that $|T \cap N| \leq 1$. If $T \cap N = \emptyset$ then $T \cup \{r\}$ is a stable set in $G(A \cup R)$, and if $T \cap N = \{n_0\}$ then $(T - \{n_0\}) \cup \{r\}$ is a stable set in $A \cup R$, for every $r \in R$. It follows that in the relation in Corollary 2, the maximum is obtained only for the first component. So i) holds.
Because $A \sim N \sim R$ and $N$ is a clique it follows that $\omega(G) = \omega(G(N)) + max\{\omega(G(A)), \omega(G(R))\}$, but $\omega(G(N)) = |N|$. So ii) holds.
Because $A \sim N \sim R$ and $N$ is a clique it follows that a domination set of minimum cardinal is a set determined by any vertex in $N$. So iii) holds.

Corollary 3 implies an algorithm for the construction of a stable set of maximum cardinal and of a clique of maximum cardinal in a trivially perfect, polar graph.

*Input:* $G = (V, E)$ a connected graph satisfying conditions in Corollary 3
*Output:* A stable set $S$ with $|S| = \alpha(G)$ and a clique $Q$ with $|Q| = \omega(G)$
*begin*
$S = \emptyset, \ Q = N$
$L = \{G\}$ // $L$ is a list of graphs
*while* $(L \neq \emptyset)$
*begin*
extract an element $H$ from $L$
*if* ($H$ is complete) $\mathtt{then}$
*Return:*
$S = S \cup \{v\}, \ \forall v \in V(H)$
$Q = Q \cup N$
*else*
Determine a weakly decomposition $(A, N, R)$ for $H$
Put $[A]_H$ and the connected component of $[R]_H$ in $L$
*end*
*end*

Facility location analysis deals with the problem of finding optimal locations for one or more facilities in a given environment ( [11]). A type of problems in facility location analysis concerns the determination of a location that minimizes the maximum distance to any other location in the network.

The following centrality indices are defined in ( [11]):
the eccentricity of a vertex u is $e_G(u) = max\{d(u,v)|v \in V\}$;
the radius is $r(G) = min\{e_G(u)|u \in V\}$;
the center of a graph G is $C(G) = \{u \in V|e_G(u) = r(G)\}$.

Using Theorem 6 we obtain the following result.

**Corollary 4.** *Let* $G = (V, E)$ *be a connected, non-complete graph and* $(A, N, R)$ *a weakly decomposition with* $G(A)$ *the weakly component. If* $G$ *is both trivially perfect and polar the following hold:*
*(i)* $e_G(u) = 2$ *for* $u \in A \cup R$*; (ii)* $e_G(u) = 1$ *for* $u \in N$*; (iii)* $r(G) = 1$*; (iv)* $C(G) = N$.

# 7 Conclusions and future work

Using the weakly decomposition we have obtained polynomial time recognition algorithms for polar graphs and directly calculated the domination number, while for the stability number and for density we give polynomial algorithms.

Future work concerns on other classes of graphs, characterized by forbidden subgraphs.

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