Int. J. of Computers, Communications & Control, ISSN 1841-9836, E-ISSN 1841-9844
Vol. V (2010), No. 5, pp. 701-709

Stable Factorization of Strictly Hurwitz Polynomials

Ö. Eğecioğlu, B. S. Yarman

Ömer Eğecioğlu
Department of Computer Science
University of California, Santa Barbara
CA 93106, USA
E-mail: omer@cs.ucsb.edu

B. Siddik Yarman
Department of Electric and Electronics Engineering
College of Engineering, Istanbul University
34320 Avcilar, Istanbul, Turkey
E-mail: sbyarman@gmail.com

Abstract: We propose a stable factorization procedure to generate a strictly Hurwitz polynomial from a given strictly positive even polynomial. This problem typically arises in applications involving real frequency techniques. The proposed method does not require any root finding algorithm. Rather, the factorization process is directly carried out to find the solution of a set of quadratic equations in multiple variables employing Newton's method. The selection of the starting point for the iterations is not arbitrary, and involves interrelations among the coefficients of the set of solution polynomials differing only in the signs of their roots. It is hoped that this factorization technique will provide a motivation to perform the factorization of two-variable positive function to generate scattering Hurwitz polynomials in two variables for which root finding methods are not applicable.

Keywords: Routh-Hurwitz stability, Hurwitz polynomial, stable factorization, Newton's method.

1 Introduction

In many microwave communication system design, modeling and simulation problems, description of lossless two ports in one or two kinds of elements is essential [5]. In the design of microwave matching networks, amplifiers or in modeling passive one port devices such as antennas, lossless two ports are either described in terms of driving point immitance or reflectance functions [6,7]. The methods known as Real Frequency Techniques (RFT) are excellent tools for design and modeling [5,8]. Once the independent descriptive parameters are selected, numerical implementations of real frequency techniques demands the construction of strictly Hurwitz polynomials. For example, in the simplified real frequency technique (SRFT), the numerator polynomial $h(p) = h_0 + h_1 p + \cdots + h_n p^n$ of the driving point input reflectance $S_{11}(p) = \frac{h(p)}{g(p)}$ completely specifies the scattering parameters of the lumped element reciprocal lossless two port as follows:

$$S_{12} = S_{21} = \frac{f(p)}{g(p)} \quad \text{and} \quad S_{22} = \frac{f(p)}{f(-p)} \frac{h(-p)}{g(p)} \tag{1}$$

provided that the monic-polynomial $f(p)$ which is constructed on the transmission zeros of the system under consideration, is pre-selected. In this representation, the denominator polynomial $g(p) = g_0 + g_1 p + \cdots + g_n p^n$ is generated as a strictly Hurwitz polynomial from the equation

$$G(p^2) = g(p)g(-p) = h(p)h(-p) + f(p)f(-p) = G_0 + G_1 p^2 + \cdots + G_n p^{2n} \tag{2}$$

which is obtained by means of the lossless condition. Once $f(p)$ is selected, (2) is specified in terms of the real coefficients $\{h_0, h_1, \ldots, h_n\}$ of $h(p)$. For many practical problems, it may be sufficient to choose $f(p)$ as $f(p) = p^k, k \leq n$. In this case, (2) results in a set of quadratic equations such that

$$\begin{aligned}
G_0 &= g_0^2 = h_0^2 \\
G_1 &= -g_1^2 + 2g_0g_2 = -h_1^2 + 2h_0h_2 \\
&\vdots \\
G_i &= (-1)^i g_i^2 + 2(g_{2i}g_0 + \sum_{j=2}^{i} (-1)^j g_{j-1}g_{2j-i+1}) = (-1)^i h_i^2 + 2(h_{2i}h_0 + \sum_{j=2}^{i} (-1)^j h_{j-1}h_{2j-i+1}) \\
&\vdots \qquad\qquad (3)\\
G_k &= G_{(i=k)} + (-1)^k \\
&\vdots \\
G_n &= (-1)^n g_n^2 = (-1)^n h_n^2
\end{aligned}$$

It should be mentioned that the general form of $f(p)f(-p)$ may be described as

$$F(p^2) = f(p)f(-p) = F_0 + F_1p^2 + \cdots + F_np^{2n}. \tag{4}$$

Then it is straightforward to revise (4.3) with the help of (4.4). At this point it is the crucial issue to generate $g(p)$ as a strictly Hurwitz polynomial either employing (2) or (4.3). If one employs (2), it is sufficient to find the roots of $G(p^2)$ and then, construct $g(p)$ on the left halfplane roots of $G(p^2)$, yielding $g(p) = g_0 + g_1p + \cdots + g_np^n$. This has been the common practice of the SRFT. However, if the problem under consideration demands the construction of lossless two-ports with two kinds of elements, then there is no way to carry out the computation by means of root finding techniques. In this case, one has to rewrite (2) in two variables as

$$G(p, \lambda) = g(p, \lambda)g(-p, -\lambda)$$

and revise (4.3) accordingly. Eventually one needs to solve (4.3) to generate $g(p, \lambda)$ as a "two variable scattering Hurwitz polynomial" [1,2]. In this representation the complex variable $p = \sigma + j\omega$ is associated with first kind of elements and the complex variable $\lambda = \Sigma + j\Omega$ is associated with the second kind of elements of the lossless two-port. Actually, this way of posing the problem may be understood as the factorization of the two variable polynomial $G(p, \lambda)$ as $g(p, \lambda)g(-p, -\lambda)$, which in turn yields the scattering Hurwitz polynomial $g(p, \lambda)$. Based on the knowledge of the authors, there is no explicit solution for the factorization of two variable polynomials in the current literature. However, for the single variable case, root finding techniques provide excellent results as described within SRFT. Therefore, in this paper, to provide an insight to the general factorization problem, an attempt will be made to come up with a numerical procedure to solve (4.3) which is specified in single variable, with the hope that the numerical procedure presented in this paper may be extended to cover the two variable factorization case.

2 Mathematical problem statement

Let $G(z^2) = G_0 + G_1z^2 + G_2z^4 + \cdots + G_nz^{2n}$ be a real polynomial with $G_0 > 0$. Consider a factorization of G of the form

$$G(z^2) = g(z)g(-z) \tag{5}$$

for a real polynomial $g(z) = g_0 + g_1 z + g_2 z^2 + \cdots + g_n z^n$ as required in (4.3). Call (5) a *stable factorization* of G, if the polynomial g is stable: that is, the real parts of the zeros of g are strictly negative. We also refer to a stable polynomial as *strictly Hurwitz.* From physical considerations that give rise to the problem, $G_0, G_1, \ldots, G_n$ are such that G admits a stable factorization. Our aim is to determine the coefficients of $g(z)$ as a function of $G_0, G_1, \ldots, G_n$.

2.1 On root finding

This one dimensional problem is theoretically solvable quite easily by root finding: Since G is a real polynomial, it can be factored as

$$G(z^2) = c(z^2 - \alpha_1)(z^2 - \alpha_2) \cdots (z^2 - \alpha_n)$$

with $c > 0$ and the α_i complex. For $i = 1, 2, \ldots, n$, let $\beta_i = \pm\sqrt{\alpha_i}$, where the sign is picked so that β_i has a negative real part. Then $g(z) = \sqrt{c}(z - \beta_1)(z - \beta_2) \cdots (z - \beta_n)$, and the g_i can be computed from this product. However, we wish to avoid this approach as the real motivation behind the treatment of the one variable case is the factorization problem in two variables to generate scattering Hurwitz polynomials, for which root finding techniques do not apply.

2.2 Basic elements of Routh-Hurwitz stability

The conditions for a real polynomial

$$g(z) = g_0 + g_1 z + g_2 z^2 + \cdots + g_n z^n \tag{6}$$

with $g_0 > 0$ to be strictly Hurwitz are given in terms of the positivity of the Hurwitz determinants

$$\Delta_i = \det \begin{bmatrix} g_1 & g_3 & g_5 & \cdots & g_{2i-1} \\ g_0 & g_2 & g_4 & \cdots & g_{2i-2} \\ 0 & g_1 & g_3 & \cdots & g_{2i-3} \\ 0 & g_0 & g_2 & \cdots & g_{2i-4} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & g_i \end{bmatrix}.$$

The indices in each row increase by two and the indices in each column decrease by one. The term g_j is taken to be zero if $j < 0$ or $j > n$. Note that $\Delta_1 = g_1$.

Theorem 1. (Routh-Hurwitz stability) A necessary and sufficient condition that the polynomial (6) is strictly Hurwitz is that $\Delta_1, \Delta_2, \ldots, \Delta_n$ be all positive [3].

Since $\Delta_n = g_n \Delta_{n-1}$, the condition that Δ_{n-1} and Δ_n be positive is equivalent to the requirement that Δ_{n-1} and g_n be positive. Furthermore, a necessary condition for (6) to be strictly Hurwitz is that all coefficients g_0 through g_n be positive.

3 The main quadratic system

Comparing coefficients in (5), we derive a quadratic system of $n+1$ equations in the variables $g_0, g_1, \ldots, g_n$:

$$G_k = \sum_{i+j=2k} (-1)^i g_i g_j, \qquad (k = 0, 1, \ldots n) \tag{7}$$

This is the system we are aiming to solve in the factorization problem. The additional constraint is that the polynomial (6) is stable. When $n = 5$,

$$\begin{aligned} G_0 &= g_0^2 \\ G_1 &= -g_1^2 + 2g_0g_2 \\ G_2 &= g_2^2 + 2g_0g_4 - 2g_1g_3 \\ G_3 &= -g_3^2 - 2g_1g_5 + 2g_2g_4 \\ G_4 &= g_4^2 - 2g_3g_5 \\ G_5 &= -g_5^2 \end{aligned} \tag{8}$$

So in this case the stable factorization problem is to find a solution $(g_0, g_1, g_2, g_3, g_4, g_5)$ of the quadratic system (8) in which each $g_i > 0$, and in addition the constraints

$$\Delta_2 = \det \begin{bmatrix} g_1 & g_3 \\ g_0 & g_2 \end{bmatrix} > 0, \quad \Delta_3 = \det \begin{bmatrix} g_1 & g_3 & 0 \\ g_0 & g_2 & g_4 \\ 0 & g_1 & g_3 \end{bmatrix} > 0, \quad \Delta_4 = \det \begin{bmatrix} g_1 & g_3 & g_5 & 0 \\ g_0 & g_2 & g_4 & 0 \\ 0 & g_1 & g_3 & g_5 \\ 0 & g_0 & g_2 & g_4 \end{bmatrix} > 0$$

are satisfied. In the general case $G_0, G_1, \ldots, G_n$ with $G_0 > 0$ are given as input. The output the solution required is real $g_0, g_1, \ldots, g_n$, with $g_0 > 0$ such that $g_0, g_1, \ldots, g_n$ is a solution of the associated quadratic system (7) of $n+1$ equations and $g(z) = g_0 + g_1z + \cdots + g_nz^n$ is strictly Hurwitz. We assume that the G_k are given so that the system has a solution of the required type.

3.1 Newton's method

We consider the vector valued function $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ which has as its set of real zeros the solutions to the quadratic system (7). For $n = 5$, $f : \mathbb{R}^6 \to \mathbb{R}^6$ is $f = (f_0, f_1, \ldots, f_5)^t$ with

$$\begin{aligned} f_0 &= x_0^2 - G_0 \\ f_1 &= -x_1^2 + 2x_0x_2 - G_1 \\ f_2 &= x_2^2 + 2x_0x_4 - 2x_1x_3 - G_2 \\ f_3 &= -x_3^2 - 2x_1x_5 + 2x_2x_4 - G_3 \\ f_4 &= x_4^2 - 2x_3x_5 - G_4 \\ f_5 &= -x_5^2 - G_5 \end{aligned}$$

We compute the Jacobian matrix as

$$J_f = 2 \begin{bmatrix} x_0 & 0 & 0 & 0 & 0 & 0 \\ x_2 & -x_1 & x_0 & 0 & 0 & 0 \\ x_4 & -x_3 & x_2 & -x_1 & x_0 & 0 \\ 0 & -x_5 & x_4 & -x_3 & x_2 & -x_1 \\ 0 & 0 & 0 & -x_5 & x_4 & -x_3 \\ 0 & 0 & 0 & 0 & 0 & -x_5 \end{bmatrix}.$$

We calculate by elementary operations

$$\det(J_f) = 2^6(-x_0x_5) \det \begin{bmatrix} -x_1 & x_0 & 0 & 0 \\ -x_3 & x_2 & -x_1 & x_0 \\ -x_5 & x_4 & -x_3 & x_2 \\ 0 & 0 & -x_5 & x_4 \end{bmatrix} = 2^6(-x_0x_5) \det \begin{bmatrix} x_1 & x_3 & x_5 & 0 \\ x_0 & x_2 & x_4 & 0 \\ 0 & x_1 & x_3 & x_5 \\ 0 & x_0 & x_2 & x_4 \end{bmatrix}$$

and $\det(J_f) = 2^6(-x_0)\Delta_5$. For general n we have a similar identity relating the Jacobian of f and Δ_n as

$$\det(J_f) = 2^{n+1}(-1)^n x_0 \Delta_n. \tag{9}$$

Thus the Jacobian J_f does not vanish at $(g_0, g_1, \ldots, g_n)$ at if $(g_0, g_1, \ldots, g_n)$ corresponds to a stable $g(x)$. In other words, starting from an initial point that is close enough to the stable solution, the Jacobian of f does not vanish. Starting with an initial vector $X_0 = (x_0, x_1, \ldots, x_n)^t$ we compute the iterates by Newton's method as

$$X_{n+1} = X_n - J_f^{-1}(X_n) f(X_n)$$

until successive iterates are within a given tolerance. The invertibility of J_f at the point X_n is guaranteed for X_n close to a stable solution $(g_0, g_1, \ldots, g_n)$. However a real solution $g(z)$ of the quadratic system found by Newton's method is not necessarily strictly Hurwitz. The polynomial we want is obtained from $g(z)$ by flipping the sign of some of its roots and making each one have negative real part, even though we do not have access to the roots themselves.

Example 1. Suppose $G(x^2) = G_0 + G_1x^2 + G_2x^4 + G_3x^6 + G_4x^8$ with $G_0 = 9.244$, $G_1 = 72.286$, $G_2 = 217.183$, $G_3 = 296.638$, $G_4 = 155.673$. $G(x^2) = g(x)g(-x)$ where

$$g(x) = g_0 + g_1x + g_2x^2 + g_3x^3 + g_4x^4$$

with $g_0 = 3.040$, $g_1 = 2.289$, $g_2 = 12.749$, $g_3 = 4.637$, $g_4 = 12.476$ is strictly Hurwitz. Starting with the initial random vector of coefficients $(1.933, 2.008, 0.181, 0.870, 2.582)$, and tolerance 0.01, Newton's method converges to the polynomial

$$3.040 + 0.004x + 11,887x^2 - 0.003x^3 + 12.477x^4$$

of the quadratic system in 14 iterations. This polynomial is not stable. Its roots are $-0.0928 \pm 0.6956j$ and $0.0929 \pm 0.6972j$.

3.2 An auxiliary problem

The necessity of being able to "flip" the sign of certain roots of a given real polynomial as indicated above results in the following auxiliary problem:

> *Given a real polynomial* $g(x) = g_0 + g_1x + \cdots + g_nx^n$ *of degree* n *with* $g_0 > 0$*, construct the real polynomial* $h(x) = h_0 + h_1x + \cdots + h_nx^n$ *with* $h_0 = g_0$*, such that the roots of* h *are* $\pm$ *roots of* g *and* h *is strictly Hurwitz.*

If we could generate the polynomials h whose roots differ from the roots of g only in their sign, then we could test each polynomial generated by the Routh-Hurwitz criteria to see if it is stable. But there cannot be an analytic way involving radicals to do this: Consider a generic fifth degree polynomial $g(x) = g_0 + g_1x + \cdots + g_5x^5$ with $g_0 > 0$. Let r be a real root of g, and let $h(x) = h_0 + h_1x + \cdots + h_5x^5$ be the polynomial with $h_0 = g_0$, which has identical roots as $g(x)$, except for its fifth root it has $-r$ instead of r. Since g_4 and h_4 are the negative of the sums of the roots of $g(x)$ and $h(x)$ respectively, we have $r = \frac{1}{2}(h_4 - g_4)$. We can also calculate $g(x)/(x - r)$ by synthetic division and compute the roots of this quartic by radicals. Thus if there were a way of computing the coefficients of $h(x)$ from those of $g(x)$ by means of radicals, then this would allow us to express the roots of a general fifth degree polynomial by radicals.

4 Algorithmic approaches to finding $g(x)$

There are two essentially distinct approaches to find the strictly Hurwitz polynomial $g(x)$ given the input data $G_0, G_1, \ldots, G_n$ with $G_0 > 0$. Both have a random component.

A1 : Generate an initial vector $X_0 = (x_0, x_1, \ldots, x_n)^t$ and run Newton's method starting with X_0. Let the converged polynomial be $h(x)$. If $h(x)$ passes the Routh-Hurwitz criteria, then it is the strictly Hurwitz polynomial desired and we are done. If not, generate another X_0 and continue.

A2 : Generate an initial vector $X_0 = (x_0, x_1, \ldots, x_n)^t$ and run Newton's method starting with X_0. Let the converged polynomial be $h(x)$. If $h(x)$ passes the Routh-Hurwitz criteria, then it is the strictly Hurwitz polynomial desired and we are done. If not, use the coefficients of $h(x)$ to generate another X_0 and continue.

A1 is simple to implement. On the other hand the number of executions of the Newton method is fewer for A2, which essentially goes from a computed $h(x)$ to another polynomial whose roots are negatives of some of the roots of $h(x)$. We shall indicate a number of methods for A2.

Example 2. In [8], the data given for a monopole antenna is modeled using the linear interpolation technique proposed for positive real functions. We used this problem for the experimental evaluation of A1 and A2. For this model $G(x^2) = G_0+G_1x^2+G_2x^4+G_3x^6+G_4x^8$ with $G_0 = 9.244$, $G_1 = 72.286$, $G_2 = 217.183$, $G_3 = 296.638$, $G_4 = 155.673$. Employing A1, the strictly Hurwitz polynomial

$$g(x) = g_0 + g_1x + g_2x^2 + g_3x^3 + g_4x^4$$

with $g_0 = 3.040$, $g_1 = 2.289$, $g_2 = 12.749$, $g_3 = 4.637$, $g_4 = 12.476$ was found. The initial vectors $X_0 = (x_0, x_1, x_2, x_3, x_4)^t$ were generated by picking x_i independently and uniformly in the range $0 < x_i < \sqrt{G_0}$. The average number of different starting points required for the Newton method for convergence to $g(x)$ with a tolerance of 0.001 is about 8 with a standard deviation of 5.

Next we consider two properties of the family of polynomials which are solutions to (4.3) and differ only in the signs of their roots.

4.1 Selection of starting points

For A2, we use the following idea: Suppose we have two real solutions $g(x)$ and $h(x)$ to the quadratic system of equations (7). Then $G(x^2) = g(x)g(-x) = h(x)h(-x)$ where $g(x)$ and $h(x)$ have the same roots up to signs. Define $F(x) = h(x)/g(x)$. Since $h(x)/g(x) = g(-x)/h(-x)$, $F(x)$ satisfies the functional equation

$$F(x)F(-x) = 1\ . \tag{10}$$

Put $F(x) = c_0 + c_1x + c_2x^2 + \cdots$ with $c_0 = 1$. >From (10), we have

$$\begin{aligned}
1 &= c_0^2\\
0 &= -c_1^2 + 2c_2\\
0 &= c_2^2 + 2c_0c_4 - 2c_1c_3\\
0 &= -c_3^2 + 2c_0c_6 - 2c_1c_5 + 2c_2c_4\\
0 &= c_4^2 + 2c_0c_8 - 2c_1c_7 + 2c_2c_6 - 2c_3c_5\\
\vdots &= \qquad\vdots
\end{aligned}$$

The general form of the k-th equation for $k \geq 1$ is

$$0 = \sum_{i+j=2k} (-1)^i c_i c_j.$$

In this infinite system, each of $c_2, c_4, c_6, \ldots$ can be expressed in terms of the coefficients $c_1, c_3, c_5, \ldots$. In fact, we can represent c_{2k} as a polynomial in $c_1, c_3, \ldots c_{2k-1}$. >From the second equation, $c_2 = \frac{1}{2}c_1^2$. Using this with the third equation we get

$$c_4 = -\tfrac{1}{2}c_2^2 + c_1c_3 = -\tfrac{1}{8}c_1^4 + c_1c_3 \quad \text{and} \quad c_6 = \tfrac{1}{2}c_3^2 + c_1c_5 - \tfrac{1}{2}c_1^3c_3 + \tfrac{1}{16}c_1^6 \; .$$

In the general case we can write

$$c_{2k} = \tfrac{1}{2}(-1)^{k+1}c_k^2 + \sum_{i=1}^{k-1} (-1)^{i+1} c_i c_{2k-i} \; . \tag{11}$$

In (11), we repeatedly substitute the expressions obtained for the earlier coefficients with even indices, we arrive at the expression of c_{2k} in terms of $c_1, c_3, c_5, \ldots$. Therefore

$$F(x) = 1 + c_1x + \tfrac{1}{2}c_1^2x^2 + c_3x^3 + (c_1c_3 - \tfrac{1}{8}c_1^4)x^4 + c_5x^5 + (\tfrac{1}{2}c_3^2 + c_1c_5 - \tfrac{1}{2}c_1^3c_3 + \tfrac{1}{16}c_1^6)x^6 + c_7x^7 + \cdots$$

Thus $h(x) = g(x)\left(1 + c_1x + \frac{1}{2}c_1^2x^2 + c_3x^3 + (c_1c_3 - \frac{1}{8}c_1^4)x^4 + c_5x^5 + \cdots\right)$ for some real numbers $c_1, c_3, c_5, \ldots$. We can use the form of the coefficients of $F(x)$ to pick a new starting point if the solution $g(x)$ we obtain from the Newton's method fails to be stable.

For algorithm A2, we generate a new initial point $h(x)$ for Newton's method from the current computed solution $g(x) = g_0 + g_1x + \cdots + g_nx^n$ with $g_0 > 0$ by setting $h_k = \sum_{i=0}^{k} g_ic_{k-i}$ for $k = 0, 1, \ldots, n$ with $c_0 = 1$ and $0 = \sum_{i=0}^{k} g_ic_{k-i}$ for $k > n$. We then express the even indexed c_i in terms of the odd index ones. After this stage, the h_k's involve only $c_1, c_3, \ldots, c_{n-1}$ (or up to c_n if n is odd.) We pick random values for these c_i's satisfying these constraints.

Example 3. For the data in Example 2, we considered the experimental evaluation of A2. The initial vector $X_0 = (x_0, x_1, x_2, x_3, x_4)^t$ was generated by picking x_i independently and uniformly in the range $0 < x_i < \sqrt{G_0}$. Following that the algorithm jumps to the next initial vector using the ideas presented above. The average number of iterations to converge to the strictly Hurwitz polynomial within a tolerance of 0.001 was 2, with a standard deviation of 1.

4.2 A linear algebraic property

Given a real polynomial $g(x) = g_0 + g_1x + \cdots + g_nx^n$ of degree n, we briefly consider the problem of constructing a new polynomial $h(x) = h_0 + h_1x + \cdots + h_nx^n$ whose roots depend on the roots of g, without actually finding the roots themselves. Without loss of generality, $g_n = 1$.

Suppose the roots of g are $\beta_1, \ldots, \beta_n$ and the required roots of h are $p(\beta_1), \ldots, p(\beta_n)$ for some polynomial p. Consider the companion matrix of g defined by

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -g_0 & -g_1 & \cdots & -g_{n-1} \end{bmatrix}$$

The characteristic polynomial of C is $\det(xI - C) = g(x)$. Then h can be expressed in terms of only the coefficients of g as $\det(xI - p(C)) = h(x)$ without calculating the zeros $\beta_1, \ldots, \beta_n$. For example for $g(x) = g_0 + g_1 x + x^2$ with zeros β_1, β_2,

$$C = \begin{bmatrix} 0 & 1 \\ -g_0 & -g_1 \end{bmatrix}$$

and the characteristic polynomial of $C^2 - 3C$ has zeros $\beta_1^2 - 3\beta_1, \beta_2^2 - 3\beta_2$. We compute

$$C^2 - 3C = \begin{bmatrix} 0 & -2 \\ 3g_0 + g_0^2 & 3g_1 + g_1^2 \end{bmatrix}$$

and therefore

$$h(x) = \det(\begin{bmatrix} x & 2 \\ -3g_0 - g_0^2 & x - 3g_1 - g_1^2 \end{bmatrix}) = 2g_0(3 + g_0) - g_1(3 + g_1)x + x^2 \ .$$

The reason for this is that C is similar to an upper triangular matrix with $\beta_1, \ldots, \beta_n$ on the diagonal [4],

$$BCB^{-1} = \begin{bmatrix} \beta_1 & & & \\ 0 & \beta_2 & & * \\ \vdots & & \ddots & \\ 0 & \ldots & 0 & \beta_n \end{bmatrix}, \text{ so that } \ Bp(C)B^{-1} = \begin{bmatrix} p(\beta_1) & & & \\ 0 & p(\beta_2) & & * \\ \vdots & & \ddots & \\ 0 & \ldots & 0 & p(\beta_n) \end{bmatrix}.$$

Functions other than polynomials can be used for p (e.g. mixtures of exponential and certain rational functions). However for this approach to work, each β_i must be transformed by the same function p. We only want to change the sign of one of the β_i at a time.

Let $I(i)$ be the matrix that is obtained from the identity matrix by changing the ith 1 to a -1. We would like to construct the matrix $s_i(C)$ such that

$$Bs_i(C)B^{-1} = I(i)BCB^{-1}. \tag{12}$$

Then the characteristic polynomial of $s_i(C)$ has zeros $\beta_1, \ldots, \beta_{i-1}, -\beta_i, \beta_{i+1}, \ldots, \beta_n$. >From (12), $s_i(C) = BI(i)B^{-1}C$. If $b_1, \ldots, b_n$ are the column vectors of B and $c_1, \ldots, c_n$ are the row vectors of B^{-1}, then $s_i(C)$ and C are related by

$$s_i(C) = (I - 2b_i c_i)C \ . \tag{13}$$

We do not need the matrix B exactly (this may involve finding eigenvalues, which are not permitted in this approach). The characteristic polynomial of the perturbed matrix in (13) will be used as a starting point for the next iteration of Newton's method, so b_i and c_i and the outer product $b_i c_i$ can be approximate.

5 Conclusions and further work

We have proposed a stable factorization procedure generate strictly Hurwitz polynomial from a given strictly positive even polynomial. The factorization process is carried out directly to find the solution of a set of quadratic equations in many variables employing Newton's method.

It is hoped that the method presented in this paper generalizes to two-variable polynomials. This would make possible the generation of scattering Hurwitz polynomials, which are the two-dimensional analogues of strict Hurwitz polynomials.

Bibliography

[1] Aksen A., “Design of Lossless Two-ports with Mixed, Lumped and Distributed Elements for Broadband Matching," PhD. Dissert., Lehrstuhl Für Nachrichtentechnik, Ruhr Universitaet Bochum, 1994.

[2] Fettweis A., “On the Scattering Matrix and the Transfer Scattering Matrix of Multi Dimensional Lossless Two-ports," Int. J. of Communication, vol. 36, pp. 374-381.

[3] Henrici P., *Applied and Computational Complex Analysis*, Vol. II, Wiley, New York, 1977.

[4] Lang S., *Linear Algebra*, Addison-Wesley, Reading, MA, 1966, p. 183.

[5] Yarman B. S., “Broadband Networks," Wiley Encyclopedia of Electrical and Electronics Engineering, Vol. II, pp. 589-604, 1999.

[6] Yarman B. S. and Aksen A., “A Reflectance-based Computer Aided Modeling Tool for High Speed/High Frequency Communication Systems," Proc. IEEE-ISCAS 2001; 4, pp. 270-273.

[7] Yarman B. S., Aksen A. and Kilinc A., “Immitance Data Modeling via Linear Interpolation Techniques," Proc. IEEE-ISCAS 2002; 3, pp. 527-530.

[8] Yarman B. S., Kilinc A. and Aksen A., “Immitance Data Modeling via Linear Interpolation Techniques: a Classical Circuit Theory Approach," Int. J. of Circuit Theory and Applications, 2004; 32, pp. 537-563.

[9] Yarman B. S. and Carlin H. J., “A Simplified Real Frequency Technique Applied to Broadband Multi-stage Amplifiers," IEEE Trans. MTT; 30, pp. 2216-2222, 1982.