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Tense θ-valued Moisil propositional logic

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Abstract: In this paper we study the tense θ-valued Moisil propositional calculus, a logical system obtained from the θ-valued Moisil propositional logic by adding two tense operators. The main result is a completeness theorem for tense θ-valued Moisil propositional logic. The proof of this theorem is based on the representation theorem of tense θ-valued Łukasiewicz-Moisil algebras, developed in a previous paper.
Keywords: Łukasiewicz-Moisil algebras, tense Moisil logic.

1 Introduction

The first contribution to the algebraic logic of finite-valued Łukasiewicz propositional calculus is Moisil's paper [18], where n-valued Łukasiewicz algebras (named today Łukasiewicz-Moisil algebras) were introduced. According to an example given by A. Rose (1957), for $n \geq 5$ the Łukasiewicz implication cannot be defined in an n-valued Łukasiewicz-Moisil algebra. Hence, Moisil discovered a new many-valued logical system (named today Moisil logic), whose algebraic models are n-valued Łukasiewicz-Moisil algebras.

In 1969, Moisil defined the θ-valued Łukasiewicz algebras, where θ is the order type of a bounded chain. These structures extend a part of the definition of n-valued Łukasiewicz algebras, but they differ from these by accepting many negation operations ([3], [10], [16], [23]). The logic corresponding to the θ-valued Łukasiewicz-Moisil algebras was developed by Boicescu [1] and Filipoiu [10] (see also [2]). This logical system is called the θ-valued Moisil propositional logic. The chrysippian endomorphisms of θ-valued Łukasiewicz-Moisil algebras are reflected in the syntax of the θ-valued Moisil propositional logic by chrysippian operations.

This paper is devoted to the tense θ-valued Moisil propositional calculus, a logical system obtained from the θ-valued Moisil propositional calculus by adding the tense operators G and H. The algebraic basis of this logic consists of tense θ-valued Łukasiewicz-Moisil algebras (tense LM_θ-algebras), algebraic structures studied in our paper [7]. We extend some of the results of [8], where a tense n-valued propositional logic was studied. The tense θ-valued Moisil propositional calculus unifies two logical systems: the classical tense logic and the θ-valued Moisil logic. The connection between these logics is realized by axioms that express the behaviour of the tense operators with respect to the chrysippian operations.

The paper is organized as follows.

In Section 2 we recall some definitions and basic facts on θ-valued Łukasiewicz-Moisil algebras and θ-valued Moisil logic, with emphasis on the connectives $\rightarrow_k$ and $\leftrightarrow_k$ and their algebraic counterparts. Section 3 deals with tense θ-valued Łukasiewicz-Moisil algebras (tense LM_θ-algebras), algebraic structures obtained from θ-valued Łukasiewicz-Moisil algebras by adding the two tense operators G and H. Section 4 contains the syntactical construction of the tense θ-valued Moisil propositional calculus. We establish some properties regarding the inferential structure of this logical system.

The Lindenbaum-Tarski algebra associated with the tense θ-valued Moisil propositional calculus is studied in Section 5. We obtain the structure of tense LM_θ-algebra. The syntactical properties of the tense θ-valued Moisil logic are reflected in this tense LM_θ-algebra, thus we use the algebraic framework in order to obtain results for the logical system.

In section 6 we define the interpretations of tense θ-valued Moisil propositional calculus and the k-tautologies of this logic. Our main result is the completeness theorem proved in this section (Theorem 26). Its proof uses the representation theorem of tense LM_θ-algebras applied to the Lindenbaum-Tarski algebra constructed in the previous section.

2 θ-valued Moisil logic and θ-valued Łukasiewicz-Moisil algebras

Let $(I, \leq)$ be a totally ordered set, with first and last element, denoted by 0 and 1 respectively, and of order type θ, through this paper.

We fix an element $k \in I$, through this paper.

In this section, we recall the θ-valued Moisil logic $\mathcal{M}_\theta$ described in [2]. The axiomatization of θ-valued Moisil propositional calculus uses the system of axioms of θ-valued calculus introduced by Boicescu [4] and Filipoiu [10]. The basic results are taken from Filipoiu [10](see also [2]). The alphabet of $\mathcal{M}_\theta$ has the following primitive symbols: an infinite set V of propositional variables; the logical connectives $\vee, \wedge, \varphi_i, \overline{\varphi}_i$ for all $i \in I$ and the parantheses (,). The set $\mathsf{Prop}(V)$ of propositions of $\mathcal{M}_\theta$ is defined by canonical induction. For each $i \in I$, we shall use the following abbreviations: $p \to_i q = \overline{\varphi}_i p \vee \varphi_i q$ and $p \leftrightarrow_i q = (p \to_i q) \wedge (q \to_i p)$. The θ-valued propositional calculus has the following *k-axioms*:

(2.1) $p \to_k (q \to_k p)$,

(2.2) $(p \to_k (q \to_k r)) \to_k ((p \to_k q) \to_k (p \to_k r))$,

(2.3) $p \wedge q \to_k p$,

(2.4) $p \wedge q \to_k q$,

(2.5) $(p \to_k q) \to_k ((p \to_k r) \to_k (p \to_k q \wedge r))$,

(2.6) $p \to_k p \vee q$,

(2.7) $q \to_k p \vee q$,

(2.8) $(p \to_k q) \to_k ((r \to_k q) \to_k (p \vee r \to_k q))$,

(2.9) $\varphi_i(p \wedge q) \leftrightarrow_k \varphi_i p \wedge \varphi_i q$, for every $i \in I$,

(2.10) $\overline{\varphi}_i(p \vee q) \leftrightarrow_k \overline{\varphi}_i p \wedge \overline{\varphi}_i q$, for every $i \in I$,

(2.11) $\varphi_j p \leftrightarrow_k \varphi_i \varphi_j p$, for every $i, j \in I$,

(2.12) $\varphi_j p \leftrightarrow_k \overline{\varphi}_i \overline{\varphi}_j p$, for every $i, j \in I$,

(2.13) $\overline{\varphi}_j p \leftrightarrow_k \varphi_i \overline{\varphi}_j p$, for every $i, j \in I$,

(2.14) $\overline{\varphi}_j p \leftrightarrow_k \overline{\varphi}_i \varphi_j p$, for every $i, j \in I$,

(2.15) $\varphi_i p \to_k \varphi_j p$, for every $i, j \in I$, $i \leq j$.

The notion of formal proof in $\mathcal{M}_\theta$ is defined in terms of the above k-axioms and the k-*modus ponens* inference rule: $\dfrac{p,\ p \to_k q}{q}$.
For briefness, we will say "modus ponens" (m.p) instead of "k-modus ponens" from now on.
We shall denote by $\vdash_k p$ that p is a *k-theorem*.
We remind some k-theorems of $\mathcal{M}_\theta$, which will be used in our proofs.

Proposition 1. *([2], p. 491, Example 3.12) The following propositions are* k*-theorems of* $\mathcal{M}_\theta$*:*

(2.16) $\vdash_k p \to_k p$,

(2.17) $\vdash_k p \leftrightarrow_k \varphi_k p$,

(2.18) $\vdash_k (\varphi_i p \vee \overline{\varphi}_i p)$, *for every* $i \in I$, $j \in I$,

(2.19) $\vdash_k (\varphi_j(p \vee q) \leftrightarrow_k \varphi_j p \vee \varphi_j q)$, $j \in I$,

(2.20) $\vdash_k (\overline{\varphi}_j(p \wedge q) \leftrightarrow_k \overline{\varphi}_j p \vee \overline{\varphi}_j q)$, $j \in I$,

(2.21) $\vdash_k ((p \to_k q) \to_k (\overline{\varphi}_k q \to_k \overline{\varphi}_k p))$,

(2.22) $\dfrac{p}{\varphi_j p}$, $j \geq k$,

(2.23) $\dfrac{\varphi_k p \to_k \varphi_k q}{p \to_k q}$.

Proposition 2. *The following propositions are* k*-theorems of* $\mathcal{M}_\theta$*:*

(2.24) $\vdash_k p \to_k (q \to_k (p \wedge q))$,

(2.25) $\vdash_k (p \wedge q \to_k r) \to_k (p \to_k (q \to_k r))$,

(2.26) $\vdash_k (p \to_k (q \to_k r)) \to_k ((p \wedge q) \to_k r)$,

(2.27) $\vdash_k (p \to_k q) \to_k ((q \to_k r) \to_k (p \to_k r))$,

(2.28) $\vdash_k (p \to_k q) \to ((r \to_k t) \to_k (p \wedge r \to_k q \wedge t))$.

Proof: *We shall establish only the* k*-theorems (2.24), (2.25) and (2.28).*

(2.24) We shall use (2.5), (2.16), (2.1), modus ponens and the Deduction Theorem (see [2], p. 495, Proposition 3.17).

$\{p, q\} \vdash_k (p \to_k p) \to_k ((p \to_k q) \to_k (p \to_k p \wedge q))$ *(2.5)*
$\{p, q\} \vdash_k p \to_k p$ *(2.16)*
$\{p, q\} \vdash_k (p \to_k q) \to_k (p \to_k p \wedge q))$ *(m.p)*
$\{p, q\} \vdash_k q \to_k (p \to_k q)$ *(2.1)*
$\{p, q\} \vdash_k q$
$\{p, q\} \vdash_k p \to_k q$ *(m.p)*
$\{p, q\} \vdash_k p \to_k (p \wedge q)$ *(m.p)*
$\{p, q\} \vdash_k p$
$\{p, q\} \vdash_k p \wedge q$ *(m.p)*
$\{p\} \vdash_k q \to_k (p \wedge q)$ *(Deduction Theorem)*
$\vdash_k p \to_k (q \to_k (p \wedge q))$ *(Deduction Theorem)*

(2.25) We shall apply (2.24), modus ponens and the Deduction Theorem.

$\{p \wedge q \rightarrow_k r, p, q\} \vdash_k p \rightarrow_k (q \rightarrow_k (p \wedge q))$ *(2.24)*
$\{p \wedge q \rightarrow_k r, p, q\} \vdash_k p$
$\{p \wedge q \rightarrow_k r, p, q\} \vdash_k q \rightarrow_k (p \wedge q)$ *(m.p)*
$\{p \wedge q \rightarrow_k r, p, q\} \vdash_k q$
$\{p \wedge q \rightarrow_k r, p, q\} \vdash_k p \wedge q$ *(m.p)*
$\{p \wedge q \rightarrow_k r, p, q\} \vdash_k p \wedge q \rightarrow_k r$
$\{p \wedge q \rightarrow_k r, p, q\} \vdash_k r$ *(m.p)*
$\{p \wedge q \rightarrow_k r, p\} \quad \vdash_k q \rightarrow_k r$ *(Deduction Theorem)*
$\{p \wedge q \rightarrow_k r\} \quad \vdash_k p \rightarrow_k (q \rightarrow_k r)$ *(Deduction Theorem)*
$\vdash_k (p \wedge q \rightarrow_k r) \rightarrow_k (p \rightarrow_k (q \rightarrow_k r))$ *(Deduction Theorem)*

(2.28) We shall use k*-axioms (2.3), (2.4), modus ponens,* k*-theorem (2.24) and the Deduction Theorem.*

$\{p \rightarrow_k q, r \rightarrow_k t, p \wedge r\} \vdash_k p \wedge r$
$\{p \rightarrow_k q, r \rightarrow_k t, p \wedge r\} \vdash_k p \wedge r \rightarrow_k p$ *(2.3)*
$\{p \rightarrow_k q, r \rightarrow_k t, p \wedge r\} \vdash_k p$ *(m.p)*
$\{p \rightarrow_k q, r \rightarrow_k t, p \wedge r\} \vdash_k p \rightarrow_k q$
$\{p \rightarrow_k q, r \rightarrow_k t, p \wedge r\} \vdash_k q$ *(m.p)*
$\{p \rightarrow_k q, r \rightarrow_k t, p \wedge r\} \vdash_k p \wedge r \rightarrow_k r$ *(2.4)*
$\{p \rightarrow_k q, r \rightarrow_k t, p \wedge r\} \vdash_k r$ *(m.p)*
$\{p \rightarrow_k q, r \rightarrow_k t, p \wedge r\} \vdash_k r \rightarrow_k t$
$\{p \rightarrow_k q, r \rightarrow_k t, p \wedge r\} \vdash_k t$ *(m.p)*
$\{p \rightarrow_k q, r \rightarrow_k t, p \wedge r\} \vdash_k q \rightarrow_k (t \rightarrow_k (q \wedge t))$ *(2.24)*
$\{p \rightarrow_k q, r \rightarrow_k t, p \wedge r\} \vdash_k t \rightarrow_k (q \wedge t)$ *(m.p)*
$\{p \rightarrow_k q, r \rightarrow_k t, p \wedge r\} \vdash_k q \wedge t$ *(m.p)*
Applying the Deduction Theorem three times we obtain that
$\vdash_k (p \rightarrow_k q) \rightarrow ((r \rightarrow_k t) \rightarrow_k (p \wedge r \rightarrow_k q \wedge t))$.

The rest of the proof is straightforward. □

The θ-valued Łukasiewicz-Moisil algebras constitute the algebraic counterpart of the θ-valued Moisil logic. The Lindenbaum-Tarski algebra of the θ-valued Moisil propositional calculus is an θ-valued Łukasiewicz-Moisil algebra (see [2], p. 500, Theorem 3.30).

We shall recall the definition of θ-valued Łukasiewicz-Moisil algebras.

Definition 3. *A* θ*-valued Łukasiewicz-Moisil algebra (*LM_θ*-algebra) is an algebra* $\mathcal{L} = (L, \wedge, \vee, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, 0_L, 1_L)$ *of type* $(2, 2, \{1\}_{i\in I}, \{1\}_{i\in I}, 0, 0)$ *such that for all* $x, y \in L$,

(2.29) $(L, \wedge, \vee, 0_L, 1_L)$ *is a bounded distributive lattice,*

(2.30) φ_i *is a bounded distributive lattice endomorphism for all* $i \in I$,

(2.31) $\varphi_i x \wedge \overline{\varphi}_i x = 0_L$; $\varphi_i x \vee \overline{\varphi}_i x = 1_L$ *for all* $i \in I$,

(2.32) $\varphi_i \circ \varphi_j = \varphi_j$ *for all* $i, j \in I$,

(2.33) If $i \leq j$ *then* $\varphi_i \leq \varphi_j$ *for all* $i, j \in I$,

(2.34) If $\varphi_i x = \varphi_i y$ *for all* $i \in I$*, then* $x = y$ *(this is known as Moisil's determination principle).*

Let $\mathcal{L} = (L, \wedge, \vee, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, 0_L, 1_L)$ be an LM_θ-algebra.
We say that $\mathcal{L}$ is *complete* if the lattice $(L, \wedge, \vee, 0_L, 1_L)$ is complete. $\mathcal{L}$ is *completely chrysippian* if for every $\{x_k\}_{k\in K}$ ($x_k \in L$ for all $k \in K$) such that $\bigwedge\limits_{k\in K} x_k$ and $\bigvee\limits_{k\in K} x_k$ exist, the following properties hold: $\varphi_i(\bigwedge\limits_{k\in K} x_k) = \bigwedge\limits_{k\in K} \varphi_i x_k$, $\varphi_i(\bigvee\limits_{k\in K} x_k) = \bigvee\limits_{k\in K} \varphi_i x_k$ $(\forall i \in I)$.

Example 4. *Let* $\mathcal{B} = (B, \wedge, \vee, ^-, 0_B, 1_B)$ *be a Boolean algebra.*
The set $D(B) = B^{[I]} = \{f | f : I \to B, i \leq j \Rightarrow f(i) \leq f(j)\}$ *of all increasing functions from* I *to* B *can be made into a* LM_θ*-algebra* $D(\mathcal{B}) = (D(B), \wedge, \vee, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, 0_{D(B)}, 1_{D(B)})$ *where* $0_{D(B)}, 1_{D(B)} : I \to B$ *are defined by* $0_{D(B)}(i) = 0_B$ *and* $1_{D(B)}(i) = 1_B$ *for every* $i \in I$*, the operations of the lattice* $(D(B), \wedge, \vee, 0_{D(B)}, 1_{D(B)})$ *are defined pointwise (cf. [2], p.6, Example 1.10) and* $(\varphi_i f)(j) = f(i)$*,* $(\overline{\varphi}_i f)(j) = (f(i))^-$ $(\forall j \in I)$ $(\forall i \in I)$*.*

Let $\mathcal{L} = (L, \wedge, \vee, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, 0_L, 1_L)$ be an LM_θ-algebra. For each $j \in I$ we consider the binary operation $\to_j$ on $\mathcal{L}$ defined by (2.35) $a \to_j b = \bar{\varphi}_j a \vee \varphi_j b = (\varphi_j a \wedge \bar{\varphi}_j b)^-$ for all $a, b \in L$. This implication is associated to $\wedge$ (like for Boolean algebras), but like for Boolean algebras also, there exists the following implication: $a \rightsquigarrow_j b = \bar{\varphi}_j a \wedge \varphi_j b$, associated to $\vee$.
The notion of morphism of LM_θ-algebras is defined as usual ([2]). Of course, a morphism of LM_θ-algebras preserves the operation $\to_j$.

3 Tense θ-valued Łukasiewicz-Moisil algebras

In this section we shall recall some definitions and basic results on tense θ-valued Łukasiewicz-Moisil algebras from [7].

Definition 5. *A tense* LM_θ*-algebra is a triple* $\mathcal{A}_t = (\mathcal{A}, G, H)$*, where* $\mathcal{A} = (A, \wedge, \vee, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, 0_A, 1_A)$ *is an* LM_θ*-algebra and* G*,* $H : A \to A$ *are two unary operations on* A *such that for all* $x, y \in A$*,*

(3.1) $G(1_A) = 1_A, H(1_A) = 1_A$*,*

(3.2) $G(x \wedge y) = G(x) \wedge G(y)$*,* $H(x \wedge y) = H(x) \wedge H(y)$*,*

(3.3) $G \circ \varphi_i = \varphi_i \circ G$*,* $H \circ \varphi_i = \varphi_i \circ H$*, for any* $i \in I$*,*

(3.4) $G(x) \vee y = 1_A$ *iff* $x \vee H(y) = 1_A$*.*

Definition 6. *Let* $(\mathcal{A}, G, H)$ *be a tense* LM_θ*-algebra. For any* $i \in I$*, let us consider the unary operations* P_i*,* F_i *defined by* $P_i x = \overline{\varphi}_i H \overline{\varphi}_i x$ *and* $F_i x = \overline{\varphi}_i G \overline{\varphi}_i x$*, for any* $x \in A$*.*

Proposition 7. *Let* $\mathcal{A} = (A, \wedge, \vee, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, 0_A, 1_A)$ *be an* LM_θ*-algebra and* G, H *be two unary operations on* A *that satisfy conditions* (3.1)*,* (3.2) *and* (3.3)*. Then, the condition* (3.4) *is equivalent with* (3.4′) $\varphi_i \leq G \circ P_i$ *and* $\varphi_i \leq H \circ F_i$ *for all* $i \in I$*.*

Thus, if we replace in Definition 5 the axiom (3.4) with the condition (3.4′), we obtain an equivalent definition of tense LM_θ-algebra.

Proposition 8. *Let* $\mathcal{A} = (A, \wedge, \vee, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, 0_A, 1_A)$ *be an* LM_θ*-algebra and* G, H *be two unary operations on* A *that satisfy conditions* (3.1) *and* (3.3)*. Then, the condition* (3.2) *is equivalent to* (3.2′) $G(a \to_k b) \leq G(a) \to_k G(b)$*;* $H(a \to_k b) \leq H(a) \to_k H(b)$ *for all* $k \in I$ *where* $\to_k$ *is defined by* (2.35)*.*

Thus, if in Definition 5 we replace the axiom (3.2) by (3.2'), we obtain an equivalent definition for tense LM_θ-algebra.

Definition 9. *A frame is a pair* (X, R)*, where* X *is a nonempty set and* R *is a binary relation on* X.

Let (X, R) be a frame and $\mathcal{L} = (L, \wedge, \vee, \{\varphi_i\}_{i\in I}, \{\bar{\varphi}_i\}_{i\in I}, 0_L, 1_L)$ be a complete and completely chrysippian LM_θ-algebra. L^X has a canonical structure of LM_θ-algebra. Let's us define for all $p \in L^X$ and $x \in X$: $G^*(p)(x) = \bigwedge\{p(y)|y \in X, xRy\}$, $H^*(p)(x) = \bigwedge\{p(y)|y \in X, yRx\}$.

Proposition 10. *For any frame* (X, R)*,* $(\mathcal{L}^X, G^*, H^*)$ *is a tense* LM_θ*-algebra.*

Let $(\mathcal{B}, G, H)$ be a tense Boolean algebra. We define on $D(B)$ the unary operations $D(G)$ and $D(H)$ by: $D(G)(f) = G \circ f$, $D(H)(f) = H \circ f$ for all $f \in D(B)$.

Lemma 11. *If* $(\mathcal{B}, G, H)$ *is a tense Boolean algebra then* $(D(\mathcal{B}), D(G), D(H))$ *is a tense* LM_θ*-algebra.*

Theorem 12. *(The representation theorem for tense* LM_θ*-algebras) For every tense* LM_θ*-algebra* $(\mathcal{A}, G, H)$ *there exist a frame* (X, R) *and an injective morphism of tense* LM_θ*-algebras* $\alpha : A \to (D(L_2))^X$*, where* $L_2 = \{0, 1\}$*, the standard Boolean algebra.*

4 Tense θ-valued Moisil logic (the syntax)

In this section we introduce the tense θ-valued Moisil propositional calculus $\mathcal{TM}_\theta$, a logical system obtained from the θ-valued propositional calculus (see [2]) by adding the two tense operators G and H. We define the notion of k-theorem and k-deduction then we establish some syntactical properties of $\mathcal{TM}_\theta$.
The alphabet of $\mathcal{TM}_\theta$ has the following primitive symbols: an infinite set V of propositional variables; the logical connectives $\vee, \wedge, \varphi_i, \overline{\varphi}_i$ for all $i \in I$; the tense operators G and H and parantheses (,). The set E of propositions of $\mathcal{TM}_\theta$ is defined by canonical induction.

Definition 13. *We shall use the following abbreviations: for all* $\alpha, \beta \in E$ *and* $i \in I$*, we define* $\alpha \to_i \beta = \overline{\varphi}_i\alpha \vee \varphi_i\beta$*;* $\alpha \leftrightarrow_i \beta = (\alpha \to_i \beta) \wedge (\beta \to_i \alpha)$*;* $F_i\alpha = \overline{\varphi}_i G \overline{\varphi}_i \alpha$*;* $P_i\alpha = \overline{\varphi}_i H \overline{\varphi}_i \alpha$.

Definition 14. *We call a k-axiom of tense* θ*-valued Moisil propositional calculus a proposition of one of the following forms:*

(4.1) The k-axioms of θ*-valued Moisil propositional calculus ((2.1)-(2.15) in Section 2);*

(4.2) $G(\alpha \to_k \beta) \to_k (G\alpha \to_k G\beta)$*;* $H(\alpha \to_k \beta) \to_k (H\alpha \to_k H\beta)$*;*

(4.3) $G\varphi_i\alpha \leftrightarrow_k \varphi_i G\alpha$*;* $H\varphi_i\alpha \leftrightarrow_k \varphi_i H\alpha$*, for all* $i \in I$*;*

(4.4) $\varphi_i\alpha \to_k GP_i\alpha$*;* $\varphi_i\alpha \to_k HF_i\alpha$*, for all* $i \in I$.

The notion of formal *k-proof* in $\mathcal{TM}_\theta$ is defined in terms of the above axioms and the following inference rules: $\dfrac{\alpha, \alpha \to_k \beta}{\beta}$ (modus ponens); $\dfrac{\alpha}{G\alpha}$ $\dfrac{\alpha}{H\alpha}$ (Temporal Generalizations)

Definition 15. *We say that a proposition* α *is a k-theorem of* $\mathcal{TM}_\theta$ *if there exists a k-proof of it. We will denote by* $\vdash_k \alpha$ *the fact that* α *is a k-theorem of* $\mathcal{TM}_\theta$.

Definition 16. *Let* $\Gamma \subseteq E$ *and* $\alpha \in E$*. We say that* α *is a k-deduction from* Γ *and write* $\Gamma \vdash_k \alpha$ *if there exist* $n \in \mathbb{N} = \{0, 1, 2, ...\}$ *and* $\alpha_1, ..., \alpha_n \in \Gamma$ *such that* $\vdash_k \bigwedge_{i=1}^{n} \alpha_i \to_k \alpha$.

We remark that the logical structure of $\mathcal{TM}_\theta$ (k-theorems and k-deduction) combines the logical stuctures of two logical systems: the θ-valued Moisil logic and tense classical logic. Further we shall prove some syntactical properties.

Lemma 17. *Let $\Gamma \subseteq E$ and $\alpha \in E$. Then $\Gamma \vdash_k \alpha$ iff there exist $n \in \mathbb{N}$ and $\alpha_1, ..., \alpha_n \in \Gamma$ such that $\vdash_k \alpha_1 \to_k (\alpha_2 \to_k ...(\alpha_n \to_k \alpha)...)$.*

Proof: *By Definition 16 and k-theorems (2.25) and (2.26).* □

Lemma 18. *Let $\Gamma \subseteq E$ and $\alpha \in E$. Then $\Gamma \vdash_k \alpha$ iff there exists $\Gamma' \subseteq \Gamma$, Γ' finite, such that $\Gamma' \vdash_k \alpha$.*

Proof: *By Definition 16 and Lemma 17.* □

Proposition 19. *Let $\Gamma, \Sigma \subseteq E$ and $\alpha, \beta \in E$. The following properties hold:*
(i) If $\vdash_k \alpha$ then $\Gamma \vdash_k \alpha$;
(ii) If $\Gamma \subseteq \Sigma$ and $\Gamma \vdash_k \alpha$ then $\Sigma \vdash_k \alpha$;
(iii) If $\alpha \in \Gamma$ then $\Gamma \vdash_k \alpha$;
(iv) $\{\alpha\} \vdash_k \beta$ iff $\vdash_k \alpha \to_k \beta$;
(v) If $\Gamma \vdash_k \alpha$ and $\{\alpha\} \vdash_k \beta$ then $\Gamma \vdash_k \beta$;
(vi) If $\Gamma \vdash_k \alpha$ and $\Gamma \vdash_k \alpha \to_k \beta$ then $\Gamma \vdash_k \beta$;
(vii) $\Gamma \vdash_k \alpha \wedge \beta$ iff $\Gamma \vdash_k \alpha$ and $\Gamma \vdash_k \beta$.

Proof: *(i) Using Definition 16 for $n = 0$. (ii) By applying Definition 16.*
(iii) Using k-theorem (2.16) and Definition 16.
(iv) We assume that $\vdash_k \alpha \to_k \beta$. Then, by Definition 16, we obtain that $\{\alpha\} \vdash_k \beta$. Conversely, if $\{\alpha\} \vdash_k \beta$ then there exists $n \in \mathbb{N}$ such that $\vdash_k (\underbrace{\alpha \wedge ... \wedge \alpha}_{n}) \to_k \beta$. By using k-axioms (2.4) and (2.5), we get that $\vdash_k (\underbrace{\alpha \wedge ... \wedge \alpha}_{n}) \leftrightarrow_k \alpha$, so $\vdash_k \alpha \to_k \beta$.
(v) We suppose that $\Gamma \vdash_k \alpha$ and $\{\alpha\} \vdash_k \beta$. Then there exist $n \in \mathbb{N}$ and $\alpha_1, .., \alpha_n \in \Gamma$ such that $\vdash_k \bigwedge_{i=1}^{n} \alpha_i \to_k \alpha$. Using (iv), it follows that $\vdash_k \alpha \to_k \beta$ and by applying k-theorem (2.27) and modus ponens, we obtain that $\vdash_k \bigwedge_{i=1}^{n} \alpha_i \to_k \beta$, so $\Gamma \vdash_k \beta$.
(vi) Let $\Gamma \vdash_k \alpha$ and $\Gamma \vdash_k \alpha \to_k \beta$. By applying Lemma 18, there exist Γ_1, $\Gamma_2 \subseteq \Gamma$ such that $\Gamma_1 \vdash_k \alpha$ and $\Gamma_2 \vdash_k \alpha \to_k \beta$. By (ii), it follows that $\Gamma_1 \cup \Gamma_2 \vdash_k \alpha$ and $\Gamma_1 \cup \Gamma_2 \vdash_k \alpha \to_k \beta$. If we consider $\Gamma_1 \cup \Gamma_2 = \{\gamma_1, ..., \gamma_n\}$, we obtain that $\vdash_k \bigwedge_{i=1}^{n} \gamma_i \to_k \alpha$ and $\vdash_k \bigwedge_{i=1}^{n} \gamma_i \to_k (\alpha \to_k \to_k \beta)$. By applying k-axiom (2.2) and modus ponens, we get that $\vdash_k \bigwedge_{i=1}^{n} \gamma_i \to_k \beta$, so $\Gamma \vdash_k \beta$.
(vii) We assume that $\Gamma \vdash_k \alpha \wedge \beta$. By using k-axioms (2.3) and (2.4) and applying (i) and (vi), we obtain that $\Gamma \vdash_k \alpha$ and $\Gamma \vdash_k \beta$. Conversely, we assume that $\Gamma \vdash_k \alpha$ and $\Gamma \vdash_k \beta$. By using k-theorem (2.24) and (i), we obtain that $\Gamma \vdash_k \alpha \to_k (\beta \to_k \alpha \wedge \beta)$. By applying twice (vi), we get $\Gamma \vdash_k \alpha \wedge \beta$. □

Theorem 20. *(The deduction theorem) Let $\Gamma \subseteq E$ and $\alpha, \beta \in E$. Then $\Gamma \cup \{\alpha\} \vdash_k \beta$ iff $\Gamma \vdash_k \alpha \to_k \beta$.*

Proof: *We assume that $\Gamma \cup \{\alpha\} \vdash_k \beta$. Then there exist $n \in \mathbb{N}$ and $\alpha_1, ..., \alpha_n \in \Gamma$ such that $\vdash_k (\bigwedge_{i=1}^{n} \alpha_i \wedge \alpha) \to_k \beta$. By applying k-theorem (2.25) and modus ponens, it follows that*

$\vdash_k \bigwedge_{i=1}^{n} \alpha_i \to_k (\alpha \to_k \beta)$. *Using Definition 16, we obtain that* $\Gamma \vdash_k \alpha \to_k \beta$. *Conversely, we suppose that* $\Gamma \vdash_k \alpha \to_k \beta$. *Thus, by Proposition 4.1 (ii), we get* $\Gamma \cup \{\alpha\} \vdash_k \alpha \to_k \beta$. *Also, by Proposition 4.1 (iii), we have that* $\Gamma \cup \{\alpha\} \vdash_k \alpha$, *hence by applying Proposition 4.1 (vi), it results that* $\Gamma \cup \{\alpha\} \vdash_k \beta$. □

Proposition 21. *In* $\mathcal{TM}_\theta$, *the following properties hold:*

(4.5) If $\vdash_k \alpha \leftrightarrow_k \beta$, *then* $\vdash_k G\alpha \leftrightarrow_k G\beta$,

(4.6) $\vdash_k G(\alpha \wedge \beta) \leftrightarrow_k (G\alpha \wedge G\beta)$.

Proof: (4.5) *By using* k*-axioms* (2.3), (2.4), k*-theorem* (2.24) *and modus ponens, we obtain that:* $\vdash_k \alpha \leftrightarrow_k \beta$ *iff* $\vdash_k \alpha \to_k \beta$ *and* $\vdash_k \beta \to_k \alpha$. *Applying the temporal generalization rule* G, *we get that* $\vdash_k G(\alpha \to_k \beta)$ *and* $\vdash_k G(\beta \to_k \alpha)$. *Then, by* k*-axiom* (4.2) *and modus ponens, it follows that* $\vdash_k G\alpha \to_k G\beta$ *and* $\vdash_k G\beta \to_k G\alpha$, *hence* $\vdash_k G\alpha \leftrightarrow_k G\beta$.

(4.6) *We shall prove that* $\vdash_k G(\alpha \wedge \beta) \to_k (G\alpha \wedge G\beta)$ *and* $\vdash_k (G\alpha \wedge G\beta) \to_k G(\alpha \wedge \beta)$. *By applying Proposition 21 (4.5) for* k*-axioms (2.3), (2.4), we obtain that* $\vdash_k G(\alpha \wedge \beta) \to_k G\alpha$ *and* $\vdash_k G(\alpha \wedge \beta) \to_k G\beta$. *Using* k*-axiom (2.5) and modus ponens, it results that* $\vdash_k G(\alpha \wedge \beta) \to_k (G\alpha \wedge G\beta)$. *By* k*-teorem (2.24) and the temporal generalization rule* G, *we obtain that* $\vdash_k G(\alpha \to_k (\beta \to_k \alpha \wedge \beta))$. *Applying* k*-axiom (4.2), modus ponens and* k*-theorem (2.27), it follows that* $\vdash_k G\alpha \to_k (G\beta \to_k G(\alpha \wedge \beta))$. *Using* k*-theorem (2.26) and modus ponens, we get that* $\vdash_k (G\alpha \wedge G\beta) \to_k G(\alpha \wedge \beta)$. *Thus* $\vdash_k G(\alpha \wedge \beta) \leftrightarrow_k (G\alpha \wedge G\beta)$. □

We remark that there exists a similar Proposition concerning H.

5 The k-Lindenbaum-Tarski algebra of tense θ-valued Moisil logic

In this section we shall prove that the k-Lindenbaum-Tarski algebra of $\mathcal{TM}_\theta$ is a tense θ-valued Łukasiewicz-Moisil algebra. Therefore, the tense θ-valued Łukasiewicz-Moisil algebras constitute the algebraic structures of $\mathcal{TM}_\theta$ and the properties of tense LM_θ-algebras reflect the syntactical properties of $\mathcal{TM}_\theta$.
We consider the binary relation $\sim_k$ on the set of all propositions E, defined by: $\alpha \sim_k \beta$ iff $\vdash_k \varphi_i\alpha \leftrightarrow_k \varphi_i\beta$ for all $i \in I$.

Lemma 22. $\sim_k$ *is an equivalence relation on* E.

For any proposition $\alpha \in E$, we denote by $[\alpha]_k$ the equivalence class of α. We can define the following operations on the set $E/_{\sim_k}$: $[\alpha]_k \vee [\beta]_k = [\alpha \vee \beta]_k$; $[\alpha]_k \wedge [\beta]_k = [\alpha \wedge \beta]_k$; $\varphi_i[\alpha]_k = [\varphi_i\alpha]_k$; $\overline{\varphi}_i[\alpha]_k = [\overline{\varphi}_i\alpha]_k$ for all $i \in I$; $G([\alpha]_k) = [G\alpha]_k$; $H([\alpha]_k) = [H\alpha]_k$; $0_k = [\overline{\varphi}_k\alpha]_k$, $1_k = [\varphi_k\alpha]_k$, where α is a k-theorem of $\mathcal{TM}_\theta$.

Proposition 23. $(E/_{\sim_k}, \wedge, \vee, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, 0_k, 1_k, G, H)$, *the* k*-Lindenbaum-Tarski algebra of* $\mathcal{TM}_\theta$, *is a tense* LM_θ*-algebra.*

Proof: *By ([2], p.500, Theorem 3.30), we have that* $(E/_{\sim_k}, \wedge, \vee, \{\varphi_i\}_{i\in I}, \{\overline{\varphi}_i\}_{i\in I}, 0_k, 1_k)$ *is an* LM_θ*-algebra. What is left to prove is that the operations* G *and* H *are well defined and the conditions (3.1)-(3.4) are satisfied. Due to the symmetrical position of* G *and* H *we shall only include the proofs for* G. *Let* $\alpha, \beta \in E$ *such that* $\alpha \sim_k \beta$. *Thus,* $\vdash_k \varphi_i\alpha \leftrightarrow_k \varphi_i\beta$ *for all* $i \in I$. *Applying Proposition 21 (4.5), we obtain that* $\vdash_k G\varphi_i\alpha \leftrightarrow_k G\varphi_i\beta$ *for all* $i \in I$. *Using* k*-axiom (4.3), it follows that* $\vdash_k \varphi_i G\alpha \leftrightarrow_k \varphi_i G\beta$ *for all* $i \in I$, *so* $G\alpha \sim_k G\beta$.

(3.1) *We have to prove that* $G([\varphi_k\alpha]_k) = [\varphi_k\alpha]_k$ *i.e. by definition of* $\sim_k$ *that* $\vdash_k \varphi_i G\varphi_k\alpha \leftrightarrow_k \varphi_i\varphi_k\alpha$ *for every* α *such that* $\vdash_k \alpha$ *and for all* $i \in I$. *Let* $\alpha \in E$ *such that* $\vdash_k \alpha$ *and* $i \in I$. *By* k*-theorem (2.22), we obtain that* $\vdash_k \varphi_k\alpha$ *and by applying the temporal generalization rule G, we obtain that* $\vdash_k G\varphi_k\alpha$. *Using* k*-axiom (2.1) and modus ponens, it results that* $\vdash_k \varphi_k\alpha \to_k G\varphi_k\alpha$ *and* $\vdash_k G\varphi_k\alpha \to_k \varphi_k\alpha$. *Thus, we get that (i)* $\vdash_k \varphi_k\alpha \leftrightarrow_k G\varphi_k\alpha$. *Using* k*-axiom (2.11), we have that (ii)* $\vdash_k \varphi_i\varphi_k\alpha \leftrightarrow_k \varphi_k\alpha$ *and by using Proposition 21(4.5), we obtain that (iii)* $\vdash_k G\varphi_i\varphi_k\alpha \leftrightarrow_k G\varphi_k\alpha$. *Using* k*-axiom (4.3) and the conditions (i),(ii), (iii), it results that* $\vdash_k \varphi_i G\varphi_k\alpha \leftrightarrow_k \varphi_i\varphi_k\alpha$.

(3.2) *Let* $\alpha, \beta \in E$. *We must prove that* $G([\alpha]_k \wedge [\beta]_k) = G([\alpha]_k) \wedge G([\beta]_k)$ *i.e.* $G(\alpha \wedge \beta) \sim_k G\alpha \wedge G\beta$ *which is equivalent with* $\vdash_k \varphi_i G(\alpha \wedge \beta) \leftrightarrow_k \varphi_i(G\alpha \wedge G\beta)$ *for all* $i \in I$. *Let* $i \in I$. *By using Proposition 21(4.6) for* $\alpha = \varphi_i\alpha$ *and* $\beta = \varphi_i\beta$, *we obtain that (i)* $\vdash_k G(\varphi_i\alpha \wedge \wedge\varphi_i\beta) \leftrightarrow_k (G\varphi_i\alpha \wedge G\varphi_i\beta)$. *By using* k*-axiom (2.9) and Proposition 21(4.5), we get that (ii)* $\vdash_k G\varphi_i(\alpha \wedge \beta) \leftrightarrow_k G(\varphi_i\alpha \wedge \varphi_i\beta)$. *By conditions (i) and (ii), we obtain that (a)* $\vdash_k G\varphi_i(\alpha\wedge\beta) \leftrightarrow_k (G\varphi_i\alpha\wedge G\varphi_i\beta)$. *By* k*-axiom (4.3), we have:* $\vdash_k G\varphi_i\alpha \leftrightarrow_k \varphi_i G\alpha$ *and* $\vdash_k G\varphi_i\beta \leftrightarrow_k \varphi_i G\beta$. *Applying* k*-theorem (2.28), it follows that (b)* $\vdash_k (G\varphi_i\alpha \wedge G\varphi_i\beta) \leftrightarrow_k \leftrightarrow_k (\varphi_i G\alpha \wedge \varphi_i G\beta)$. *By conditions (a), (b) and* k*-axiom (4.3), we obtain that* $\vdash_k \varphi_i G(\alpha \wedge \wedge\beta) \leftrightarrow_k \varphi_i(G\alpha \wedge G\beta)$.

(3.3) *We have to prove that* $\vdash_k \varphi_j G\varphi_i\alpha \leftrightarrow_k \varphi_j\varphi_i G\alpha$ *for all* $i, j \in I$. *Let* $i, j \in I$. *By* k*-axiom (2.11), we obtain that (a)* $\vdash_k \varphi_j\varphi_i G\alpha \leftrightarrow_k \varphi_i G\alpha$. *Using* k*-axiom (4.3), we have that (b)* $\vdash_k \varphi_j G\varphi_i\alpha \leftrightarrow_k G\varphi_j\varphi_i\alpha$. *By* k*-axioms (2.11) and Proposition 21(4.5), it follows that (c)* $\vdash_k G\varphi_j\varphi_i\alpha \leftrightarrow_k G\varphi_i\alpha$. *By (a), (b), (c) and k-axiom (4.3), we get that* $\vdash_k \varphi_j G\varphi_i\alpha \leftrightarrow_k \varphi_j\varphi_i G\alpha$.

(3.4) *Since by Proposition 7, the condition (3.4) is equivalent with (3.4'), we shall prove that* $[\varphi_i\alpha]_k \leq [GP_i\alpha]_k$ *for all* $i \in I$, *i.e.* $\vdash_k \varphi_j\varphi_i\alpha \to_k \varphi_j GP_i\alpha$ *for all* $i, j \in I$. *Let* $i, j \in I$. *By* k*-axiom (2.13), we have that* $\vdash_k P_i\alpha \leftrightarrow_k \varphi_j P_i\alpha$. *Applying Proposition 21 (4.5), it follows that* $\vdash_k GP_i\alpha \leftrightarrow_k G\varphi_j P_i\alpha$. *Using* k*-axiom (4.3), it results that (1)* $\vdash_k GP_i\alpha \leftrightarrow_k \varphi_j GP_i\alpha$. *Also, by* k*-axiom (2.11), we have that (2)* $\vdash_k \varphi_i\alpha \leftrightarrow_k \varphi_j\varphi_i\alpha$. *By (1), (2) and* k*-axiom (4.4), we get that* $\vdash_k \varphi_j\varphi_i\alpha \to_k \varphi_j GP_i\alpha$.

□

6 Semantics and completeness theorem of tense θ-valued Moisil logic

This section concernes with the semantics of $\mathcal{TM}_\theta$, which combines the properties of Kripke semantics for $\mathcal{T}$ and the algebraic semantics for $\mathcal{M}_\theta$. We establish a completeness theorem for $\mathcal{TM}_\theta$ by using the representation theorem of tense θ-valued Łukasiewicz-Moisil algebras [7].

Definition 24. *Let* (X, R) *be a frame. A valuation of* $\mathcal{TM}_\theta$ *is a function* $v : E \times X \to L_2^{[I]}$ *such that for all* $\alpha, \beta \in E$ *and* $x \in X$, *the following equalities hold:* $v(\alpha \to_k \beta, x) = v(\alpha, x) \to_k v(\beta, x)$; $v(\alpha \wedge \beta, x) = v(\alpha, x) \wedge v(\beta, x)$; $v(\alpha \vee \beta, x) = v(\alpha, x) \vee v(\beta, x)$; $v(\varphi_i\alpha, x) = \varphi_i v(\alpha, x)$ *for any* $i \in I$; $v(\overline{\varphi}_i\alpha, x) = \overline{\varphi}_i v(\alpha, x)$ *for any* $i \in I$; $v(Gp, x) = \bigwedge\{v(p, y) | xRy\}$; $v(Hp, x) = \bigwedge\{v(p, y) | yRx\}$.

The first five conditions of the previous definition reflect "the many-valued past" of $\mathcal{TM}_\theta$ (see [2], p.487) and the last two conditions correspond to "the tense past" of $\mathcal{TM}_\theta$ (see [5], p.93).

Definition 25. *We say that a proposition* α *is a k-tautology and we write* $\models_k \alpha$ *if for every frame* (X, R), *for any valuation* $v : E \times X \to L_2^{[I]}$ *and for all* $x \in X$, *we have* $v(\alpha, x)(k) = 1$.

The following result establishes the equivalence between the k-theorems and the k-tautologies of $\mathcal{TM}_\theta$. The proof of the main implication is based on the representation theorem for tense θ-valued Łukasiewicz-Moisil algebras (Theorem 12).

Theorem 26. *(Completeness theorem). For any proposition α of $\mathcal{TM}_\theta$, we have: $\vdash_k \alpha$ iff $\models_k \alpha$.*

Proof: $(\Rightarrow)$. *We shall prove by induction on the definition of $\vdash_k \alpha$ that for every frame (X, R) and for any valuation $v : E \times X \to L_2^{[I]}$, we have $v(\alpha, x)(k) = 1$, for all $x \in X$.*
Let (X, R) be a frame, $v : E \times X \to L_2^{[I]}$ be a valuation and $x \in X$.
- *We suppose that α is a k-axiom.*

(a) *Let α be $G(p \to_k q) \to_k (Gp \to_k Gq)$ with $p, q \in E$. It is known that $a \to_k (b \to_k c) = (a \wedge b) \to_k c$ ([7], p.6, Proposition 2.1 (l)). We have:* $v(\alpha, x)(k) = v(G(p \to_k q) \to_k (Gp \to_k Gq), x)(k) = [v(G(p \to_k q), x) \to_k (v(Gp, x) \to_k v(Gq, x))](k) = [(v(G(p \to_k q), x) \wedge v(Gp, x)) \to_k v(Gq, x)](k) = [\bigwedge\limits_{xRy}((v(p, y) \to_k v(q, y)) \wedge v(p, y)) \to_k \to_k \bigwedge\limits_{xRy} v(q, y)](k) = [\overline{\varphi}_k \bigwedge\limits_{xRy}((v(p, y) \to_k v(q, y)) \wedge v(p, y)) \vee \varphi_k \bigwedge\limits_{xRy} v(q, y)](k) = [(\bigwedge\limits_{xRy}(v(p, y) \to_k v(q, y)) \wedge v(p, y))(k)]^- \vee (\bigwedge\limits_{xRy} v(q, y))(k) = [\bigwedge\limits_{xRy}((v(p, y)(k))^- \vee v(q, y)(k)) \wedge \wedge v(p, y)(k)]^- \vee (\bigwedge\limits_{xRy} v(q, y)(k)) = [\bigwedge\limits_{xRy}(v(q, y)(k) \wedge v(p, y)(k))]^- \vee (\bigwedge\limits_{xRy} v(q, y)(k))$.
Since $v(q, y)(k)$, $v(p, y)(k) \in L_2$ and $v(q, y)(k) \wedge v(p, y)(k) \leq v(q, y)(k)$, we obtain that $\bigwedge\limits_{xRy}(v(q, y)(k) \wedge v(p, y)(k)) \leq \bigwedge\limits_{xRy} v(q, y)(k)$. Since in a Boolean algebra we have $a \leq b$ iff $\bar{a} \vee b = 1$, we get that $[\bigwedge\limits_{xRy}(v(q, y)(k) \wedge v(p, y)(k))]^- \vee (\bigwedge\limits_{xRy} v(q, y)(k)) = 1$.

(b) *Let α be $G\varphi_i p \leftrightarrow_k \varphi_i Gp$ with $p \in E$ and $i \in I$. Then* $v(\alpha, x)(k) = v(G\varphi_i p \leftrightarrow_k \varphi_i Gp, x)(k) = v((G\varphi_i p \to_k \varphi_i Gp) \wedge (\varphi_i Gp \to_k G\varphi_i p), x)(k) = [(v(G\varphi_i p, x) \to_k v(\varphi_i Gp, x)) \wedge (v(\varphi_i Gp, x) \to_k v(G\varphi_i p, x))](k)$. *Since $L_2^{[I]}$ is complete and completely chrysippian, it follows that $v(G\varphi_i p, x) = \bigwedge\limits_{xRy} \varphi_i v(p, y) = \varphi_i(\bigwedge\limits_{xRy} v(p, y)) = v(\varphi_i Gp, x)$. We know that $a \to_k a = 1$ ([7], p.6, Proposition 2.1 (f)), hence $v(\alpha, x)(k) = 1$.*

(c) *Let α be $\varphi_i p \to_k GP_i p$ with $i \in I$. We have:* $v(\alpha, x)(k) = v(\varphi_i p \to_k GP_i p, x)(k) = (v(\varphi_i p, x) \to_k v(GP_i p, x))(k) = (\varphi_i v(p, x) \to_k \bigwedge\limits_{xRy} v(P_i p, y))(k) =$
$= (\varphi_i v(p, x) \to_k \bigwedge\limits_{xRy} \bigvee\limits_{zRy} \varphi_i v(p, z))(k) = \overline{\varphi}_k(\varphi_i v(p, x))(k) \vee \varphi_k(\bigwedge\limits_{xRy} \bigvee\limits_{zRy} \varphi_i v(p, z))(k) =$
$= [v(p, x)(i)]^- \vee \bigwedge\limits_{xRy} \bigvee\limits_{zRy} v(p, z)(i)$. *Let $y \in X$ such that xRy. Then $v(p, x)(i) \leq \bigvee\limits_{zRy} v(p, z)(i)$, hence $v(p, x)(i) \leq \bigwedge\limits_{xRy} \bigvee\limits_{zRy} v(p, z)(i)$. We obtain that $[v(p, x)(i)]^- \vee \bigwedge\limits_{xRy} \bigvee\limits_{zRy} v(p, z)(i) = 1$.*

- *We assume that α was obtained by applying the modus ponens rule. We have that $v(\beta, x)(k) = 1$ and $v(\beta \to_k \alpha, x)(k) = 1$. But* $v(\beta \to_k \alpha, x)(k) = (v(\beta, x) \to_k v(\alpha, x))(k) = (\overline{\varphi}_k v(\beta, x) \vee \varphi_k v(\alpha, x))(k) = \overline{\varphi}_k(v(\beta, x))(k) \vee \varphi_k(v(\alpha, x))(k) = [v(\beta, x)(k)]^- \vee v(\alpha, x)(k)$.
We deduce that $v(\alpha, x)(k) = 1$.

• *We suppose that* $\alpha = G\beta$ *such that* $\vdash_k \beta$. *We have that* $\nu(\beta, x)(k) = 1$, *for every* $x \in X$. *Then* $\nu(G\beta, x)(k) = (\bigwedge_{xRy} \nu(\beta, y))(k) = \bigwedge_{xRy} \nu(\beta, y)(k) = 1$.

$(\Leftarrow)$. *We shall prove that if* $\not\vdash_k \alpha$ *then* $\not\models_k \alpha$. *Assume that* $\not\vdash_k \alpha$, *so* $[\alpha]_k \neq 1_k$. *By using Proposition 23, we have that the* k*-Lindenbaum-Tarski algebra* $(E/_{\sim_k}, G, H)$ *of* $\mathcal{TM}_\theta$ *is a tense* LM_θ*-algebra. Applying the representation theorem for tense* LM_θ*-algebras (Theorem 12), there exist a frame* (X, R) *and an injective morphism of tense* LM_θ*-algebras* $d : (E/_{\sim_k}, G, H) \rightarrow$ $\rightarrow (D(L_2)^X, G^*, H^*)$. *Let us consider the function* $\nu : E \times X \rightarrow L_2^{[I]}$ *defined by* $\nu(\alpha, x) = d([\alpha]_k)(x)$, *for all* $\alpha \in E$ *and* $x \in X$. *It is straightforward to prove that* ν *is a valuation. Since* d *is injective and* $[\alpha]_k \neq 1_k$, *we obtain that* $d([\alpha]_k) \neq 1_{D(L_2)^X}$, *hence there exists* $a \in X$ *such that* $\nu(\alpha, a) = d([\alpha]_k)(a) \neq 1_{D(L_2)}$. *Thus* α *is not a* k*-tautology.* □

7 Concluding Remarks

The tense θ-valued Moisil propositional calculus $\mathcal{TM}_\theta$ can be viewed as a common generalization of the θ-valued Moisil propositional logic $\mathcal{M}_\theta$ and the classical tense logic $\mathcal{T}$.
$\mathcal{TM}_\theta$ combines the logical structures of these logical systems and its semantic is inspired from the semantics of $\mathcal{T}$ and $\mathcal{M}_\theta$. The main result of this paper is a completeness theorem for $\mathcal{TM}_\theta$. Its proof is derived from the representation theorem of tense θ-valued Łukasiewicz-Moisil algebras [7].

An open problem is to obtain a proof of the representation theorem for tense θ-valued Łukasiewicz-Moisil algebras by using Theorem 26.

The next step in the study of tense aspects of Moisil logic is to define the tense θ-valued predicate logic (the syntax and the semantic) and the algebras corresponding of this logic (polyadic tense θ-valued Łukasiewicz-Moisil algebras). We hope to prove a completeness theorem for tense θ-valued Moisil predicate logic and a representation theorem for the corresponding algebras. The tense logics corresponding to the LM_θ-algebras with negations [16] will be the subject of another paper.

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