

Direct Method for Stability Analysis of Fractional Delay Systems

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Abstract: In this paper, a direct method is presented to analyze the stability of fractional order systems with single and multiple commensurate time delays. Using the approach presented in this study, first, without using any approximation, the transcendental characteristic equation is converted to an algebraic one with some specific crossing points. Then, an expression in terms of system parameters and imaginary root of the characteristic equation is derived for computing the delay margin. Finally, the concept of stability is expressed as a function of delay. An illustrative example is presented to confirm the proposed method results.

Keywords: Control, fractional delay systems, stability windows, Root-Locus.

1 Introduction

Time delay is an inherent part of many dynamical and physical systems. The device delay often appears in computer based control systems, the wireless and web-based control systems, communication systems and so on. Also, the existence of various sensors and actuators in the feedback loop is the cause of delay in many systems. Moreover, taking an automatic computer-based control system or a process control with networked transmission for example, it is necessary and more reasonable to simultaneously consider the possible transmission delay for the corresponding control law and the device delay due to computing or processing. Recently, much attention has been paid to the subjects of stability, stabilization and control of the time delay systems [1] - [5]. Fractional order models would be more accurate than integer order models. In fact, real world processes generally or most likely are fractional order systems.

In a generic sense, the fractional-order systems are identified by non-integer powers of the Laplace variable s . When time delays and fractional-order derivative are involved in dynamical systems, we have fractional-delay systems. The stability of fractional-delay systems can be determined by the root location of the characteristic equations. The most natural way to find the location of the roots of a linear fractional order system is to solve its corresponding characteristic equation. But in our case this will be a transcendental one, being thus generally impossible to solve it directly. For this reason, most of the existing approaches study stability of such systems by finding the crossings of poles through the imaginary axis [8]. There has been a large effort to deal with this problem, as can be seen by the large quantity of articles dealing with it for the standard case (integer order systems); see [5], and others.

In [6], the necessary and sufficient conditions for the BIBO stability of fractional order delay systems have been introduced. From the numerical analysis point of view, the effective numerical algorithms have been discussed in [7] and [8] for the evaluation of BIBO stability of fractional

order delay systems. In [9] a heavy computation scheme based on the Cauchy's integral has been proposed to test the stability of such systems, and in [10], a technique based on the Lambert W function was used for the same purpose. In this work, an analytical direct approach to determine all possible stability regions in the parametric space of delay is proposed. The original idea in this strategy is derived based on the method reported in [5] to achieve the stability criteria of integer order delay systems. The proposed method is an analytically elegant procedure that first converts the transcendental characteristic equation into an algebraic equation without the transcendentality by eliminating the highest degree of commensurability terms successively. The resulting algebraic equation without the transcendentality also enables us to easily determine the delay dependency of the system stability and the sensitivities of crossing roots (root tendency) with respect to the time delay.

2 Preliminaries and Definitions

A general class of fractional order LTI (linear time invariant) systems with multiple commensurate delays of retarded type is taken into account:

$$D_t^{1/\alpha} x(t) = \frac{d^{1/\alpha} x(t)}{dt^{1/\alpha}} = A_0 + \sum_{\ell=1}^n A_\ell x(t - \ell\tau)$$

The fractional delay characteristic equation of above system can be expressed in the general form of

$$C(\sqrt[\alpha]{s}, \tau) = \det \left(s^{1/\alpha} I - A_0 - \sum_{\ell=1}^n A_\ell e^{-\ell\tau s} \right) = \sum_{\ell=0}^n P_\ell(\sqrt[\alpha]{s}) e^{-\ell\tau s} = 0 \quad (1)$$

where parameter τ is non-negative, such that $\tau \in \mathbb{R}^+$ and $P_\ell(\sqrt[\alpha]{s})$ for $\ell \in \mathbb{N}_N$ is a real polynomial in the complex variable $\sqrt[\alpha]{s}$ (where $\alpha \in \mathbb{N}$). Note that the zeros of characteristic equation (1) are in fact the poles of the system under investigation. We find out from [6] that the transfer function of a system with a characteristic equation in the form of (1) will be H_∞ stable if, and only if, it doesn't have any pole at $\Re(s) \geq 0$. For fractional order systems, if an auxiliary variable of $v = \sqrt[\alpha]{s}$ is used, a practical test for the evaluation of stability can be obtained. By applying this auxiliary variable in characteristic equation (1), the following relation is obtained:

$$C_v(v, \tau) = \sum_{\ell=0}^n P_\ell(v) e^{-\ell\tau v^\alpha} \quad (2)$$

This will transform the domain of the system from a multisheeted Riemann surface into the complex plane, where the poles can be easier calculated. In this new variable, the instability region of the original system is not given by the right half-plane, but in fact by the region described as:

$$|\angle v| \leq \frac{\pi}{2\alpha} \quad (3)$$

with $v \in \mathbb{C}$. Let us assume that $s = \pm j\omega$ or in other words, $s = \omega e^{\pm j\pi/2}$ are the roots of characteristic equation (1) for a $\tau \in \mathbb{R}^+$. Then for the auxiliary variable, the roots are defined as follows:

$$v = \sqrt[\alpha]{s} = \sqrt[\alpha]{\omega} e^{\pm j\pi/2\alpha} \quad (4)$$

3 Crossing position

The main objective of this section is to present a new method for the evaluation of stability and determination of the unstable roots of a fractional delay system. The proposed method eliminates the transcendental term of the characteristic equation without using any approximation and converts it into a equation without the transcendentality such that its real roots coincide with the imaginary roots of the characteristic equation exactly. If the characteristic equation (1) has a solution of $s = j\omega_c$ then $C(\sqrt[\alpha]{-s}, \tau) = 0$ will have the same solution.

$$C(\sqrt[\alpha]{-s}, \tau) = \sum_{\ell=0}^n P_{\ell}(\sqrt[\alpha]{-s}) e^{\ell\tau s} = 0 \tag{5}$$

Characteristic equation (5) can be written in terms of the auxiliary parameter \bar{v} as:

$$C(\bar{v}, \tau) = \sum_{\ell=0}^n P_{\ell}(\bar{v}) e^{\ell\tau v^{\alpha}} = 0 \tag{6}$$

A recursive procedure should be developed to achieve that purpose. Therefore, let us define

$$C^{(1)}(v, \tau) = \sum_{\ell=0}^{n-1} [P_0(\bar{v})P_{\ell}(v) - P_n(v)P_{n-\ell}(\bar{v})] e^{-\ell\tau v^{\alpha}} = \sum_{\ell=0}^{n-1} P_{\ell}^{(1)}(v) e^{-\ell\tau v^{\alpha}} = 0 \tag{7}$$

Then, we have

$$C^{(1)}(\bar{v}, \tau) = \sum_{\ell=0}^{n-1} [P_0(v)P_{\ell}(\bar{v}) - P_n(\bar{v})P_{n-\ell}(v)] e^{\ell\tau v^{\alpha}} = \sum_{\ell=0}^{n-1} P_{\ell}^{(1)}(\bar{v}) e^{\ell\tau v^{\alpha}} = 0 \tag{8}$$

Where

$$P_{\ell}^{(1)}(v) = P_0(\bar{v})P_{\ell}(v) - P_n(v)P_{n-\ell}(\bar{v}) \tag{9}$$

We can easily repeat this procedure to eliminate commensuracy terms successively by defining a new polynomial

$$P_{\ell}^{(r+1)}(v) = P_0^{(r)}(\bar{v})P_{\ell}^{(r)}(v) - P_{n-r}^{(r)}(v)P_{n-r-\ell}^{(r)}(\bar{v}) \tag{10}$$

and an augmented characteristic equation

$$C^{(r)}(v, \tau) = \sum_{\ell=0}^{n-r} P_{\ell}^{(r)}(v) e^{-\ell\tau v^{\alpha}} = 0 \tag{11}$$

By repeating this procedure n times, we eliminate the highest degree of commensuracy terms and obtain the following augmented characteristic equation

$$C^{(n)}(v) = P_0^{(n)}(v) = 0 \tag{12}$$

Where

$$P_0^{(n)}(v) = P_0^{(n-1)}(\bar{v})P_0^{(n-1)}(v) - P_1^{(n-1)}(v)P_1^{(n-1)}(\bar{v}) \tag{13}$$

It should be emphasized that that if $s = j\omega_c$ is the solution of (1) for some τ , then it is also a solution of (12). If we substitute $\bar{v} = (e^{-j\pi/\alpha})v$ and $v = \sqrt[\alpha]{\omega_c} e^{j\pi/2\alpha}$ in (13), we get the following equation in ω

$$D(\omega) = \left(P_0^{(n-1)}(e^{-j\pi/\alpha} v) P_0^{(n-1)}(v) - P_1^{(n-1)}(v) P_1^{(n-1)}(e^{-j\pi/\alpha} v) \right) \Big|_{v=\sqrt[\alpha]{\omega_c} e^{j\pi/2\alpha}} \tag{14}$$

The corresponding value of time delay is then computed by

$$\tau^* = \frac{1}{\omega_c} \tan^{-1} \left(\frac{\Im[(P_0^{(n-1)}(v))/(P_1^{(n-1)}(v))]}{\Re[-(P_0^{(n-1)}(v))/(P_1^{(n-1)}(v))]} \right) \Big|_{v=\sqrt[n]{\omega_c} e^{\frac{j\pi}{2\alpha}}} + \frac{2k\pi}{\omega_c} \quad ; k \in \mathbb{Z}^+ \tag{15}$$

Theorem 1. A system with characteristic equation (1) has finite crossing points for any $\tau \in \mathbb{R}^+$.

Proof: Let assume $s = j\omega_c$ be a pair of roots for $C(\sqrt[n]{s}, \tau)$ then $v = \sqrt[n]{s} = \sqrt[n]{\omega_c} e^{\pm j\pi/2\alpha}$ would be a pair of roots for $D(\omega)$. since $D(\omega)$ is a finite degree polynomial with maximum degree of $(P_0^{(n)}(v))$ then the number of crossing points of (1) that are the real roots of $D(\omega)$ is finite. \square

The whole ω values, for which $s = j\omega$ is a root of equation (1) for some non-negative delays, is defined as the crossing frequency set.

$$\Omega = \left\{ \omega \in \mathbb{R}^+ \mid C(\sqrt[n]{s}, \tau) = 0, \text{ for some } \tau \in \mathbb{R}^+ \right\} \tag{16}$$

Corollary 2. If the system given as (1) is stable for $\tau = 0$ (i.e. system without delay) and $\Omega = \phi$, then the system will be stable for all positive values of $\tau \in \mathbb{R}^+$.

Proof: From the fact that there are no roots crossing the imaginary axis. \square

4 Direction of crossing

After the crossing points of characteristic equation (1) from the imaginary axis are obtained, the goal now is to determine whether each of these root crossings from the imaginary axis is a stabilizing cross or a destabilizing cross. Assume that (s, τ) is a simple root of $C(\sqrt[n]{s}, \tau) = 0$. The root tendency for each ω_{cm} and τ_{mk} is defined as:

$$\text{Root Tendency} = RT|_{s=j\omega_c} = \text{sgn} \left(\Re \left(S_\tau^s \Big|_{\substack{s=j\omega_{cm} \\ \tau=\tau_{mk}}} \right) \right) = \text{sgn} \left(\Re \left(- \frac{\partial C / \partial \tau}{\partial C / \partial s} \Big|_{s=j\omega_c} \right) \right) \tag{17}$$

If it is positive, then it is a destabilizing crossing, whereas if it is negative, this means a stabilizing crossing. Notice that root tendency represents the direction of transition of the roots at $j\omega_{cm}$ as τ increases from $\tau_{mk-\epsilon}$ to $\tau_{mk+\epsilon}$, $0 < \epsilon \ll 1$.

Theorem 3. The root tendency at a crossing, $j\omega_c$ is invariant with respect to time delay τ_{mk} .

Proof: One can find $ds/d\tau$ for simple roots of (1) as follows:

$$\frac{ds}{d\tau} = - \frac{\partial C / \partial \tau}{\partial C / \partial s} = \frac{\sum_{\ell=0}^n P_\ell(\sqrt[n]{s}) \ell s e^{-\ell\tau s}}{\sum_{\ell=0}^n \frac{P_\ell(\sqrt[n]{s})}{ds} e^{-\ell\tau s} - \sum_{\ell=0}^n P_\ell(\sqrt[n]{s}) \ell \tau s e^{-\ell\tau s}} \tag{18}$$

Based on (18) and definition given in (17), the root tendency of each time delay τ_{mk} is written as follows:

$$\begin{aligned} RT|_{s=j\omega_c}^\tau &= \text{sgn} \left(\Re \left(S_\tau^s \Big|_{s=j\omega_c} \right) \right) = \text{sgn} \left(\Re \left(- \frac{\partial C / \partial \tau}{\partial C / \partial s} \Big|_{s=j\omega_c} \right) \right) \\ &= \text{sgn} \left(\Re \left(\frac{\sum_{\ell=0}^n P_\ell(\sqrt[n]{s}) \ell s e^{-\ell\tau s}}{\sum_{\ell=0}^n \frac{P_\ell(\sqrt[n]{s})}{ds} e^{-\ell\tau s} - \sum_{\ell=0}^n P_\ell(\sqrt[n]{s}) \ell \tau s e^{-\ell\tau s}} \right) \right) = \text{sgn} \left(\Re \left(\frac{\sum_{\ell=0}^n \frac{P_\ell(\sqrt[n]{s})}{ds} e^{-\ell\tau s}}{\sum_{\ell=0}^n P_\ell(\sqrt[n]{s}) \ell s e^{-\ell\tau s}} - \frac{\tau}{s} \right)^{-1} \right) \tag{19} \\ &= \text{sgn} \left(\Re \left(\frac{\sum_{\ell=0}^n \frac{P_\ell(\sqrt[n]{s})}{ds} e^{-\ell\tau s}}{\sum_{\ell=0}^n P_\ell(\sqrt[n]{s}) \ell s e^{-\ell\tau s}} \right) \right) = \text{sgn} \left(\Im \left(\frac{\sum_{\ell=0}^n \frac{P_\ell(\sqrt[n]{s})}{ds} e^{-\ell\tau s}}{\sum_{\ell=0}^n P_\ell(\sqrt[n]{s}) \ell e^{-\ell\tau s}} \right) \right) \Big|_{\substack{s=j\omega_c \\ \tau=\tau_{m1} + \frac{2k\pi}{\omega_c}}} \end{aligned}$$

The root tendency in each time delay τ_{mk} is independent from the time delay itself and constant for each crossing frequency, because $e^{-\ell\tau s}$ and $P_\ell(\sqrt[n]{s})$ do not depend on τ_{mk} . \square

5 Illustrative Example

We present an example case, which display all the features discussed in the text.

Example 4. *This example has been taken from [3] and [8]. Consider the following linear time-invariant fractional order system with one delay:*

$$C(\sqrt{s}, \tau) = (\sqrt{s})^3 - 1.5(\sqrt{s})^2 + 4(\sqrt{s}) + 8 - 1.5(\sqrt{s})^2 e^{-\tau s} \quad (20)$$

This system has a pair of poles ($s = \pm 8j$) on the imaginary axis for $\tau = 0$. A very involved calculation scheme based on Cauchy's integral has been used in [9] to show that this system is unstable for $\tau = 0.99$ and stable for $\tau = 1$. Our objective in this example is to find all the stability windows based on the method described in this article for this system. Using auxiliary variable $v = \sqrt{s}$ into (20) the characteristic equation is obtained as:

$$C_v(v, \tau) = v^3 - 1.5v^2 + 4v + 8 - 1.5v^2 e^{-\tau v^2} \quad (21)$$

By applying the criterion expressed in the previous section, we can eliminate exponential term from (21) as follows:

$$jv^6 + 1.5(1 - j)v^5 + 14(1 + j)v^3 - j16v^2 + 32(1 - j)v + 64 = 0 \quad (22)$$

By inserting expression $v = \sqrt{\omega} e^{j\frac{\pi}{4}} = a(1 + j)$ in the above equation, we get:

$$8a^6 - 12a^5 - 56a^3 + 32a^2 + 64a + 64 = 0 \quad (23)$$

Where $a = \sqrt{\omega/2}$. The real solutions of (23) for a is

$$a = 1.82, \quad v_2 = 1.82(1 + j), \quad \omega = 6.6248 \quad (24)$$

$$a = 2, \quad v_1 = 2(1 + j), \quad \omega = 8 \quad (25)$$

The corresponding infinite countable time delays of the cross points in (24) and (25) are obtained with regards to relation (15) as $\tau_{1k} = 0.0499 + 0.3019k\pi$ and $\tau_{2k} = 0.25k\pi$, respectively.

By applying the criterion expressed in the previous section, it is easy to find out that a destabilizing crossing of roots ($RT = +1$) has occurred at $\tau = 0.25k\pi$ for $s = \pm j6.6248$ and a stabilizing crossing ($RT = -1$) has taken place at $\tau = 0.0499 + 0.3019k\pi$ for $s = \pm j8$ for all values of $k \in \mathbb{Z}^+$. Therefore, we will have 5 stability windows as follows: $0.0499 < \tau < 0.7854$, $0.9983 < \tau < 1.5708$, $1.9486 < \tau < 2.3562$, $2.8953 < \tau < 3.1416$ and $3.8437 < \tau < 3.9270$ which agree with the results presented by [3] and [8]. Note that at $\tau = 3.9269$, an unstable pair of poles crosses toward the right half-plane, and before this unstable pole pair can turn to the left half-plane at $\tau = 4.7922$, another unstable pair of poles goes toward the right half-plane at $\tau = 4.7123$; and thus, the system can not recover the stability. To get a better understanding of the properties of this system, its Root-Locus curve has been plotted as a function of delay in Fig.1.

6 Conclusions

An efficient method to analyze the BIBO stability of a large class of time-delayed fractional order systems for both single and commensurate-delay cases is proposed. The method introduces an augmented equation whose real roots give the finite values of crossing frequencies at which stability feature of the system change. According to the infinitely countable time delays corresponding to each crossing point, the parametric space of τ is discretized to investigate stability in each interval. Finally, an illustrative example is presented to highlight the proposed approach.

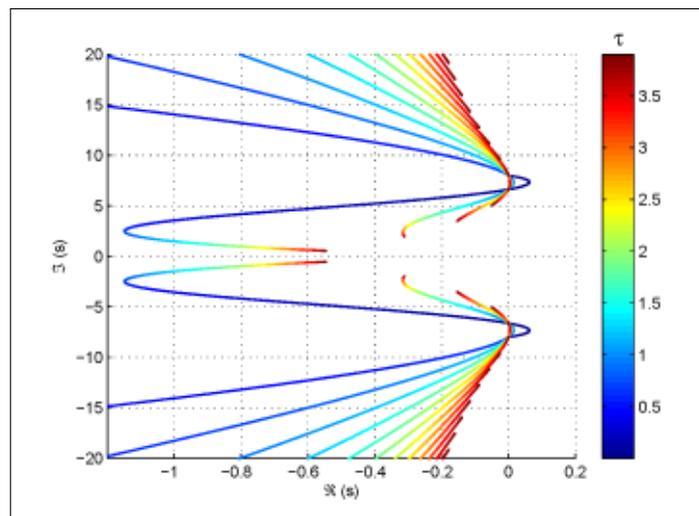


Figure 1: Root-loci for $C(\sqrt{s}, \tau)$ until $\tau=3.9$

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