

# Solving Method for Linear Fractional Optimization Problem with Fuzzy Coefficients in the Objective Function

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### Abstract:

The importance of linear fractional programming comes from the fact that many real life problems are based on the ratio of physical or economic values (for example cost/-time, cost/volume, profit/cost or any other quantities that measure the efficiency of a system) expressed by linear functions. Usually, the coefficients used in mathematical models are subject to errors of measurement or vary with market conditions. Dealing with inaccuracy or uncertainty of the input data is made possible by means of the fuzzy set theory.

Our purpose is to introduce a method of solving a linear fractional programming problem with uncertain coefficients in the objective function. We have applied recent concepts of fuzzy solution based on  $\alpha$ -cuts and Pareto optimal solutions of a bi-objective optimization problem.

As far as solving methods are concerned, the linear fractional programming, as an extension of linear programming, is easy enough to be handled by means of linear programming but complicated enough to elude a simple analogy. We follow the construction of the fuzzy solution for the linear case introduced by Dempe and Ruziyeva (2012), avoid the inconvenience of the classic weighted sum method for determining Pareto optimal solutions and generate the set of solutions for a linear fractional program with fuzzy coefficients in the objective function.

**Keywords:** fuzzy programming, fractional programming, multi-objective programming.

## 1 Introduction

In the present paper the fuzzy linear fractional optimization problem (with fuzzy coefficients in the objective function) is considered (FOLFP). The importance of linear fractional programming comes from the fact that many real life problems are based on the ratio of physical or economic values (for example cost/time, cost/volume, profit/cost or any other quantities that measure the efficiency of a system) expressed by linear functions. Usually, the coefficients used in mathematical models are subject to errors of measurement or vary with market conditions. Dealing with inaccuracy or uncertainty of the input data is made possible by means of the fuzzy set theory.

In [6] a basic introduction to the main models and methods in fuzzy linear programming is presented and, as a whole, linear programming problems with fuzzy costs, fuzzy constraints and fuzzy coefficients in the constraint matrix are analyzed.

[9] presents a brief survey of the existing works on comparing and ranking any two interval numbers on the real line and then, on the basis of this, gives two approaches to compare any two

interval numbers. [1] generalizes concepts of the solution of the linear programming problem with interval coefficients in the objective function based on preference relations between intervals and it unifies them into one general framework together with their corresponding solution methods.

Dempe and Ruziyeva [2] solved the linear programming problem with triangular fuzzy coefficients in the objective function. They derived explicit formulas for the bounds of the intervals used for defining the membership function of the fuzzy solution of linear optimization and determined all efficient points of a bi-objective linear programming problem by means of weighted sum optimization. Their approach is based on Chanas and Kuchta’s method [1] based on calculating the sum of lengths of certain intervals.

The state of the art in the theory, methods and applications of fractional programming is presented in Stancu-Minasian’s book [8]. Multiple-optimization problems are widely discussed in [4]. Many mathematical models considers multiple criteria and a large variety of solution methods are introduced in recent literature (see for instance [3, 5]). [7] introduces a linear programming approach to test efficiency in multi-objective linear fractional programming problems.

Interpreting uncertain coefficients in FOLFP as fuzzy numbers, we derive formulas for the  $\alpha$ -cut intervals that describe the fuzzy fractional objective function. Operating with intervals, we construct a bi-objective parametric linear fractional programming problem (BOLFP). We use the procedure introduced by Lotfi, Noora et al. to generate all efficient points of (BOLFP). The membership value of a feasible solution is calculated as cardinality of the set of parameters for which the feasible solution is efficient in (BOLFP). In this way, the fuzzy optimal solution is obtained as a fuzzy subset of the feasible set of (FOLFP) and the decision-maker will have the opportunity to choose the most convenient crisp solution among those with the highest membership value.

In Section 2 we formulate the fuzzy optimization problem FOLFP, set up notation and terminology, construct equivalent bi-objective linear fractional programming problem BOLFP, and describe a new way to generate efficient points for BOLFP. A procedure to compute the membership function of each feasible solution is given in Section 3. We give an example of FOLFP and its fuzzy solution obtained by applying the new solving method in Section 4. Conclusions and future works are inserted in Section 5.

## 2 Fuzzy linear fractional optimization problem

The linear fractional programming problem with fuzzy coefficients in the objective function is

$$\max_{x \in X} \frac{\tilde{c}^T x + \tilde{c}^0}{\tilde{d}^T x + \tilde{d}^0} \tag{1}$$

where  $X = \{x \in R^n | Ax \leq b, x \geq 0\}$ ,  $A$  is the  $m \times n$  matrix of the linear constraints,  $b \in R^m$  is the vector representing the right-hand-side of the constraints,  $x$  is the  $n$ -dimensional vector of decision variables,  $\tilde{c}, \tilde{d} \in R^n, \tilde{c}^0, \tilde{d}^0 \in R$  represent the fuzzy coefficients of the objective function.

### 2.1 Equivalent problems

We replace Problem (1) by Problem (2) that describes the maximization of the  $\alpha$ -cut intervals of the objective function over the initial feasible set.

$$\max_{x \in X} \left[ \frac{c_L(\alpha)^T x + c_L^0(\alpha)}{d_R(\alpha)^T x + d_R^0(\alpha)}, \frac{c_R(\alpha)^T x + c_R^0(\alpha)}{d_L(\alpha)^T x + d_L^0(\alpha)} \right] \tag{2}$$

where  $c_L(\alpha)$  ( $c_R(\alpha)$ ),  $d_L(\alpha)$  ( $d_R(\alpha)$ ),  $c_L^0(\alpha)$  ( $c_R^0(\alpha)$ ),  $d_L^0(\alpha)$  ( $d_R^0(\alpha)$ ) represent the vectors of the left sides (right sides respectively) of the  $\alpha$ -cut intervals of the fuzzy coefficients of the objective function. For a deeper discussion of  $\alpha$ -cut intervals we refer the reader to [10] and [11].

According to Chanas and Kuchta [1], an interval  $[a, b]$  is smaller than an interval  $[c, d]$  if and only if  $a \leq c$  and  $b \leq d$  with at least one strong inequality. In this way, for each fixed  $\alpha$ -cut, Problem (2) is equivalent to the bi-objective linear fractional programming problem (3)

$$\max \frac{c_L(\alpha)^T x + c_L^0(\alpha)}{d_R(\alpha)^T x + d_R^0(\alpha)}, \max \frac{c_R(\alpha)^T x + c_R^0(\alpha)}{d_L(\alpha)^T x + d_L^0(\alpha)} \quad (3)$$

subject to

$$x \in X.$$

## 2.2 The special case of triangular fuzzy numbers

Let us consider now a subclass of fuzzy numbers – the continuous triangular fuzzy that are represented by a triple  $(c_L, c_T, c_R)$  [2]. In this case  $c_L(\alpha) = \alpha c_T + (1 - \alpha) c_L$  and  $c_R(\alpha) = \alpha c_T + (1 - \alpha) c_R$ . By simple analogy, similar formulas are derived for  $d_L(\alpha)$ ,  $d_R(\alpha)$ ,  $c_L^0(\alpha)$ ,  $c_R^0(\alpha)$ ,  $d_L^0(\alpha)$  and  $d_R^0(\alpha)$ . Using these formulas, Problem (4) is constructed in order to be solved instead of Problem (3).

$$\begin{aligned} \max & \frac{(\alpha c_T + (1 - \alpha) c_L)^T x + \alpha c_T^0 + (1 - \alpha) c_L^0}{(\alpha d_T + (1 - \alpha) d_R)^T x + \alpha d_T^0 + (1 - \alpha) d_R^0} \\ \max & \frac{(\alpha c_T + (1 - \alpha) c_R)^T x + \alpha c_T^0 + (1 - \alpha) c_R^0}{(\alpha d_T + (1 - \alpha) d_L)^T x + \alpha d_T^0 + (1 - \alpha) d_L^0} \end{aligned} \quad (4)$$

subject to

$$x \in X.$$

So far, the construction is similar to the construction given in [2] for the linear case. The analogy cannot continue due to the fact that the weighted sum of linear fractional objectives is not linear fractional objective any more. That is why we formulate a new method for generating efficient points for Problem (4) in next section.

## 2.3 The generation of efficient points

Solving a multiple-objective linear fractional problem by optimizing the weighted sum of the objective functions is not impossible but it is quite complicated. Instead of that, we will construct the convex combination of the marginal solutions of Problem (4), and use each point in the combination to generate an efficient point. By our method all points on the segment that connects the two marginal solutions are mapped to the set of all Pareto optimal solutions of Problem (4). The generation method is based on a linear procedure (proposed by Lotfi, Noora et al. [7]) that determine whether a feasible point is efficient for a linear fractional programming problem or not. Theorem 1 reformulates the results introduced in [7] by applying them to the bi-objective linear fractional programming problem (3).

**Theorem 1.** (adapted from [7]) For arbitrarily fixed  $\alpha \in [0, 1]$ ,  $x^* \in X$  is a weakly efficient solution in (3) if and only if the optimal value of problem (5) below is zero.

$$\begin{aligned}
 \max \quad & t \\
 \text{s.t.} \quad & 0 \leq t \leq d_1^- + d_1^+, \\
 & 0 \leq t \leq d_2^- + d_2^+, \\
 & c_L(\alpha)^T x + c_L^0(\alpha) - d_1^+ = (c_L(\alpha)^T x^* + c_L^0(\alpha)) \theta_1, \\
 & d_R(\alpha)^T x + d_R^0(\alpha) + d_1^- = (d_R(\alpha)^T x^* + d_R^0(\alpha)) \theta_1, \\
 & c_R(\alpha)^T x + c_R^0(\alpha) - d_2^+ = (c_R(\alpha)^T x^* + c_R^0(\alpha)) \theta_2, \\
 & d_L(\alpha)^T x + d_L^0(\alpha) + d_2^- = (d_L(\alpha)^T x^* + d_L^0(\alpha)) \theta_2, \\
 & x \in X, \\
 & d_1^+, d_1^-, \theta_1, d_2^+, d_2^-, \theta_2 \geq 0.
 \end{aligned} \tag{5}$$

We have obtained the following theoretical result that will be needed in Section 3.

**Proposition 2.** *For an arbitrary fixed  $\alpha \in [0, 1]$  and for any convex combination  $x^* \in X$  of the marginal solutions of (3), the components of  $x$  in the optimal solution of (5) represent an weakly efficient point of (3).*

We give only the main ideas of the proof. We have to construct the cone that contains all points that dominate the given point on the segment that connects the two marginal solutions. One particular case is presented in Figure 1 but the basic idea of the proof can be drawn from it. Let us consider that the feasible set of Problem (3) is the quadrilateral  $ABCD$  and for a given value of  $\alpha$ , rotational points of the two objective functions are  $E$  and  $F$  respectively. Marginal solutions are  $B$  for the first objective and  $D$  for the second one.  $G$  is an arbitrary point on the segment  $BD$ . The value of the first objective function is constant on the line  $EG$  and can be improved by rotating  $EG$  anticlockwise toward  $EB$ . On  $EB$  the maximal value is reached. The value of the second objective function is constant on  $FG$  and can be improved by rotating  $FG$  clockwise toward  $FD$ . On  $FD$  the maximal value is reached. Hence, all points contained in the cone  $EGF$  dominate point  $G$ . By solving Problem (5) with starting point  $G$ , the feasible points from the cone  $EGF$  are analyzed and an efficient point from the cone is selected.

A more complete theory may be obtained by deriving conditions under which Theorem 1 can be used in generating all efficient points for a multiple-objective linear fractional programming problem.

### 3 Solving method

In this section we propose a procedure to compute the membership function of each feasible solution of Problem (1). The procedure is essentially based on the new method of generation of efficient points for the bi-objective linear fractional programming problem (3).

- For each  $\alpha \in [0, 1]$ :
  - Initialize  $\psi(\alpha) = \phi$ .
  - Find marginal solutions  $x_\alpha^1$  and  $x_\alpha^2$  in (3) and construct their convex combination  $x_\alpha(\lambda) = \lambda x_\alpha^1 + (1 - \lambda) x_\alpha^2, \lambda \in [0, 1]$ .
  - For each  $\lambda \in [0, 1]$ , solve Problem (5) with  $x^* = x_\alpha(\lambda)$ . Due to Theorem 1, an efficient point  $x_{\alpha\lambda}^{eff}$  for Problem (3) is obtained.  $\psi(\alpha) = \psi(\alpha) \cup \{x_{\alpha\lambda}^{eff}\}$ .
- For each  $x \in X$  calculate its membership value  $\mu(x) = \text{card}\{\alpha | x \in \psi(\alpha)\}$ .

Practically,  $\psi(\alpha)$  represents the set of Pareto optimal solutions of (3) for corresponding parameter  $\alpha$ . Each feasible solution of (1), for which at least one  $\alpha \in [0, 1]$  exists so that it is efficient in (3), has non-negative membership value in the fuzzy solution of (1).

### 4 Example

To complete the discussion we describe the procedure of finding fuzzy solution for one simple example.

$$\max \frac{-\tilde{1}x_1 + \tilde{10}x_2 + \tilde{4}}{\tilde{2}x_1 + \tilde{5}x_2 + \tilde{1}} \tag{6}$$

subject to

$$\begin{aligned} x_1 &\geq 1, \\ -x_1 + 2x_2 &\leq 1, \\ 2x_1 + x_2 &\leq 8, \\ 2x_2 &\geq 1, \\ x_1, x_2 &\geq 0. \end{aligned}$$

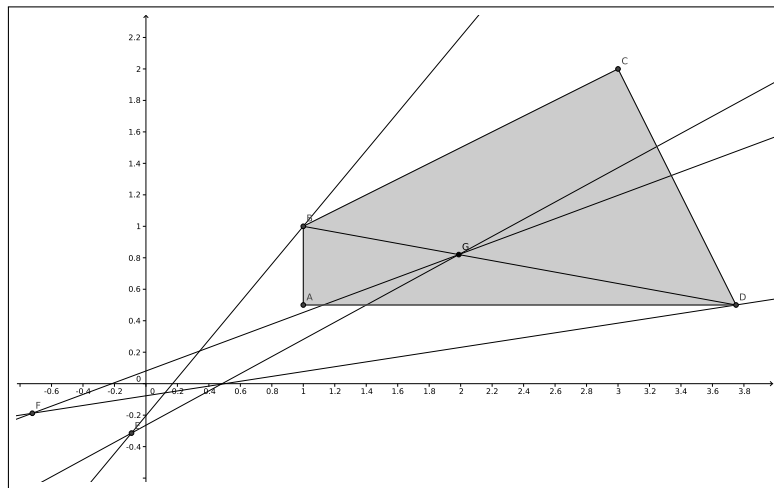


Figure 1: Feasible set of Problem (6) and efficient points for Problem (4) for  $\alpha = 0.2$

Figure 1 describes the feasible set of the problem as the quadrilateral  $ABCD$ .

First time, fuzzy coefficients were treated as triangular fuzzy numbers with parameters  $(c - 1, c, c + 5)$  and  $(c^0 - 1, c^0, c^0 + 1)$  for nominator,  $(d - 1, d, d + 2)$  and  $(d^0 - 1, d^0, d^0 + 10)$  for denominator. Two pairs of marginal solutions were found:  $x_\alpha^1 = (1, 1)$ ,  $x_\alpha^2 = (1, 0.5)$  for  $\alpha \in [0, 0.55]$ , and  $x_\alpha^1 = x_\alpha^2 = (1, 1)$  for  $\alpha \in [0.56, 1]$ . Hence,  $\mu((1, 1)) = 1$  and  $\mu(x) = 0.55$  for each  $x$  on the segment line  $[(1, 1), (1, 0.5)]$ . All other feasible solutions have membership value equal to 0 in the fuzzy optimal solution. See Table 1.

$\alpha$	$x_\alpha^1$	$x_\alpha^2$	$\psi(\alpha)$	$x$	$\mu(x)$
$[0, 0.55]$	$(1, 1)$	$(1, 0.5)$	$AB$	$x \in [AB)$	0.61
$[0.56, 1]$	$(1, 1)$	$(1, 1)$	$B$	$x = B$	1

Table 1: Computational results for the first set of fuzzy numbers

Second time, fuzzy coefficients were treated as triangular fuzzy numbers with parameters  $(c - 1, c, c + 5)$  and  $(c^0 - 1, c^0, c^0 + 1)$  for nominator,  $(d - 2, d, d + 10)$  and  $(d^0 - 1, d^0, d^0 + 10)$  for denominator. Three pairs of marginal solutions were found:  $x_\alpha^1 = (1, 1)$ ,  $x_\alpha^2 = (3.75, 0.5)$  for  $\alpha \in [0, 0.23]$ ,  $x_\alpha^1 = (1, 1)$ ,  $x_\alpha^2 = (1, 0.5)$  for  $\alpha \in [0.24, 0.62]$ , and  $x_\alpha^1 = x_\alpha^2 = (1, 1)$  for  $\alpha \in [0.63, 1]$ . Hence,  $\mu((1, 1)) = 1$ ,  $\mu(1, 0.5) = 0.62$ ,  $\mu(x) = 0.23$  for each  $x$  on the segment line  $[(3.75, 0.5), (1, 0.5)]$ , and  $\mu(x) = 0.62$  for each  $x$  on the segment line  $[(1, 1), (1, 0.5)]$ . All other feasible solutions have membership value equal to 0 in the fuzzy optimal solution. For  $\alpha = 0.2$ , the marginal solutions and the rotational points of the two objectives are presented in Figure 1 by  $B$  and  $D$  and  $E$  and  $F$  respectively. The efficient set is, in this case,  $AB \cup AD$ . See Table 2.

$\alpha$	$x_\alpha^1$	$x_\alpha^2$	$\psi(\alpha)$	$x$	$\mu(x)$
$[0, 0.23]$	$(1, 1)$	$(3.75, 0.5)$	$AB \cup AD$	$x \in (AD)$	0.23
$[0.24, 0.62]$	$(1, 1)$	$(1, 0.5)$	$AB$	$x \in (AB)$	0.61
$[0.63, 1]$	$(1, 1)$	$(1, 1)$	$B$	$x = B$	1

Table 2: Computational results for the second set of fuzzy numbers

## 5 Conclusions and future works

In the present paper the fuzzy linear fractional programming problem with fuzzy coefficients in the objective functions is considered. The calculation of the membership function of the fuzzy solution is described. The initial problem is first transformed into an equivalent  $\alpha$ -cut interval problem. Further, the problem is transformed into a bi-objective parametric linear fractional programming problem. For each value of the parameter, the bi-objective problem is solved by generating its efficient points from any convex combination of its marginal solutions. The solving method works for any kind of fuzzy numbers but particular formulas were derived for the special case of triangular fuzzy numbers.

The decision making process under uncertainty is widely studied nowadays. In our future works we will focus on identifying how the procedure of generation of efficient points for bi-objective fractional programming problems can be applied in a more general case. Also, we will study other kind of fuzzy optimization problems with fractional objectives.

## Acknowledgements

This research was partially supported by the Ministry of Education and Science, Republic of Serbia, Project numbers TR36006 and TR32013.

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