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# Direct Evolutionary Search for Nash Equilibria Detection

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**Abstract:** A Direct method of computing mixed form Nash equilibria of a normal form game by using a simple evolutionary algorithm is proposed. The Direct Evolutionary Search algorithm (DES) uses a generative relation for Nash equilibria with binary tournament selection and uniform mutation. Numerical experiments are used to illustrate the efficiency of the method.
**Keywords:** Mixed form Nash equilibria, Evolutionary algorithms, generative relation

## 1 Introduction

The problem of computing Nash equilibria of normal form games is one of the most challenging problems faced by computer scientists. The complexity of this problem - varying hugely from a type of game to another - is still studied.

Normal form games can be solved by modern heuristics by transforming them into an optimization or fixed point problem [2]. An interesting challenge consists on designing a method that solves the game directly, as a *game* and not as a corresponding optimization problem.

The main problem in directly approaching the Nash equilibria is caused by the fact that there does not exist an order (or even preorder) relation defined for game situations that can guide a search operator towards the Nash equilibrium. While any kind of optimization endeavor is driven by a corresponding order (or preorder) relation defined in the objective space, in the case of Nash equilibria the lack of such a relation limits the design possibilities of computational heuristics.

However, recently [1] a generative relation for Nash equilibria - called the Nash ascendancy relation - has been proposed. The Nash ascendancy relation is defined on strategy profiles. Even if it does not induce an actual order, numerical results indicate that it is capable of guiding a search operator towards Nash equilibria.

### Nash equilibria - definition and generative relation

Nash equilibrium [4] is the most popular solution concept in noncooperative game theory. A finite strategic game is defined by a set o players, a set of strategies available to each player and a set of payoff functions for each player and denoted by $\Gamma = (N, S, U)$ where:

- $N$ represents the set of players, $N = \{1, ...., n\}$, $n$ is the number of players;
- for each player $i \in N$, $S_i$ represents the set of actions available to him, $S_i = \{s_{i_1}, s_{i_2}, ..., s_{i_{m_i}}\}$ where $m_i$ represents the number of strategies available to player $i$ and $S = S_1 \times S_2 \times ... \times S_N$ is the set of all possible situations of the game;
- for each player $i \in N$ denote by $p_{ij}$ the probability that he selects its $j$-th action, $j \in \{1, ..., m_i\}$. Then $P_i = (p_{i1}, ..., p_{im_i})$ represents a probability distribution over the set of actions of player $i$ and $P = (P_1, ..., P_n)$ represents a mixed strategy profile for the game, where $p_{ij} \in [0, 1]$ and $\sum_{j=1}^{m_i} p_{ij} = 1$, $\forall i = \overline{1, N}$ and $\forall j = \overline{1, m_i}$;

- for each player $i \in N$, $u_i(P)$ represents the expected payoff for the mixed strategy $P$;

Denote by $(Q_i, P^*_{-i})$ the strategy profile obtained from $P^*$ by replacing the probability distribution of player $i$ with $Q_i$ i.e.

$$(Q_i, P^*_{-i}) = (P^*_1, P^*_2, ..., P^*_{i-1}, Q_i, P^*_{i+1}, ..., P^*_n).$$

A strategy profile $P \in S$ for the game $\Gamma$ represents a Nash equilibrium [2, 4] if no player has anything to gain by unilaterally changing his own strategy while the others do not modify theirs.

Several methods to compute NE of a game have been developed. For a review on computing techniques for the NE see [2] and [6].

**Nash ascendancy relation**

A generative relation for Nash equilibria is a relation between two strategy profiles that enables their comparison with respect to the Nash solution concept, i.e. it evaluates which one is "closer" to equilibrium. In [1] such a generative relation has been introduced and shown that solutions that are non-dominated/ascended with respect to this relation are exactly the Nash equilibria of the game.

Consider two mixed strategy profiles $P$ and $Q$ and the operator $\kappa$ that associates the cardinality of the set

$$\kappa(P, Q) = |\{i \in \{1, ..., n\} | u_i(Q_i, P_{-i}) \geq u_i(P), Q_i \neq P_i\}|$$

to the pair $(P, Q)$, i.e. the number of players $i$ that would benefit from unilaterally switching their strategies from $P_i$ to $Q_i$.

The operator $\kappa$ can be used to induce a relation on the set of mixed strategy profiles in the following manner: we can say that the strategy profile $P$ *Nash ascends* the strategy profile $Q$ in and we write $P \prec_N Q$ if the inequality

$$\kappa(P, Q) < \kappa(Q, P)$$

holds, i.e. there are les players that can increase their payoffs by switching their strategy from $P$ to $Q$ than vice-versa. It can be said that strategy profile $P$ is more stable (closer to equilibrium) then strategy $Q$.

Regarding the Nash ascendancy relation, two strategy profiles $P$ and $Q$ may be in the following situation:

1. $P$ ascends $Q$: $P \prec_N Q$ $(\kappa(P, Q) < \kappa(Q, P))$
2. $Q$ ascends $P$: $Q \prec_N P$ $(\kappa(P, Q) > \kappa(Q, P))$
3. or $\kappa(P, Q) = \kappa(Q, P)$ and $P$ and $Q$ are considered *indifferent* (neither $P$ ascends $Q$ nor $Q$ ascends $P$).

The strategy profile $P^*$ is called non-ascended in Nash sense (NAS) if there does not exist any mixed strategy profile $Q$ such that

$$Q \neq P^* \text{ and } Q \prec_N P^*.$$

A very important result in [1] shows that for pure strategies all non-ascended strategies are NE and also all NE are non-ascended strategies. The proof in the case of mixed strategies is similar and direct. Thus the Nash ascendancy relation can be used to characterize the equilibria of a game and can be considered as a generative relation for NEs.

The Nash ascendancy concept was introduced with the purpose to compare two strategy profiles [1] during the search of an evolutionary algorithm in order to compute NEs of a game.

Consider two strategy profiles $P^*$ and $P$ from $\Delta$. Then $k : \Delta \times \Delta \to N$ associates the pair $(P^*, P)$ the cardinality of the set

$$k(P^*, P) = card\{i \in \{1, ..., n\} | u_i(P_i, P^*_{-i}) > u_i(P^*), P_i \neq P^*_i\}.$$

This set is composed by the players $i$ that would benefit if - given the strategy profile $P^*$ - would change their strategy from $P^*_i$ to $P_i$.

It is obvious that for any $P^*, P \in S$, we have

$$0 \leq k(P^*, P) \leq n.$$

**Definition 1.** Let $P, Q \in \Delta$. We say the strategy profile $P$ Nash ascends $Q$ and we write $P \prec Q$ if the inequality

$$k(P, Q) < k(Q, P),$$

holds.

*Remark* 1.1. Two strategy profiles $P, Q \in \Delta$ can have the following relation:

1. either $P$ Nash ascends $Q$,
2. either $Q$ Nash ascends $P$,
3. if $k(P, Q) = k(Q, P)$ then $P$ and $Q$ are *indifferent*

**Definition 2.** A strategy profile $P^* \in S$ is called non-dominated with respect to the Nash ascendancy relation (NNS) if

$$\nexists Q \in \Delta, Q \neq P^* \text{such that } Q \prec P^*.$$

**Definition 3.** The set of all Nash nondominated strategy profiles with respect to the Nash ascendancy relation is the set containing all nondominated strategies i.e.

$$NND = \{s \in S | s \text{ Nash non-dominated with respect to the Nash ascendancy relation}\}$$

**Proposition 1.** A strategy profile $P^* \in \Delta$ is a NE iff the equality

$$k(P^*, Q) = 0, \forall Q \in \Delta,$$

holds.

**Proof:** Let $P^* \in \Delta$ be a NE. Suppose there exists $Q \in \Delta$ such that $k(P^*, Q) = w$, $w \in \{1, ..., n\}$. Therefore there exists $i \in \{1, ..., n\}$ such that $u_i(Q_i, P^*_{-i}) > u_i(P^*)$ and $Q_i \neq P^*_i$, which contradicts the definition of NE.

For the second implication, let $P^* \in \Delta$ such that $\forall Q \in \Delta, k(P^*, Q) = 0$. This means that for all $i \in \{1, ..., n\}$ and for any $Q_i \in \mathcal{P}_i$ we have $u_i(Q_i, P^*_{-i}) \leq u_i(P^*)$. It follows that $P^*$ is a NE. □

**Proposition 2.** All NE are Nash nondominated solutions (NND) i.e.

$$NE \subseteq NND.$$

**Proof:** Let $P^* \in \Delta$ be a NE. Suppose that there exists a strategy profile $P \in \Delta$ such that $P \prec P^*$. It follows that $k(P, P^*) < k(P^*, P)$. But $k(P^*, P) = 0$, therefore we must have $k(P, P^*) < 0$ which is not possible since $k(P, P^*)$ denotes the cardinality of a set. □

**Proposition 3.** All Nash nondominated solutions are NE, i.e.

$$NND \subseteq NE.$$

**Proof:** Let $P^*$ be a nondominated strategy profile. Suppose $P^*$ is not NE. Therefore there must exist (at least) one $i \in \{1, ..., n\}$ and a strategy $P_i \in \mathcal{P}_i$ such that

$$u_i(P_i, P^*_{-i}) > u_i(P^*),$$

holds. Let's denote by $Q = (P_i, P^*_{-i})$. It means that $k(P^*, Q) = 1$. But $k(Q, P^*) = 0$. Therefore $k(Q, P^*) < k(P^*, Q)$ which means that $Q \prec P^*$ thus the hypothesis that $P^*$ is nondominated is contradicted. □

Using propositions 2 and 3 it is obvious that the next result holds:

**Proposition 4.** The following relation holds:

$$NE = NND,$$

i.e. all NE are also Nash nondominated and also all Nash nondominated strategies are NE.

### Direct evolutionary search

The Direct Evolutionary Search (DES) algorithm is a simple evolutionary algorithm based on tournament selection and uniform mutation designed for Nash equilibria detection.

**Individuals** represent strategy profiles of the game. Real valued encoding is used. Each individual is represented as a vector composed of $(\sum_{i=1}^{n} m_i)$ components between 0 and 1. For each individual $n$ corresponding payoffs are computed.

**Selection** Binary tournament selection is used in the following manner: For each individual $i$, another one $k$ is selected randomly. If individual $k$ Nash ascends individual $i$, $k$ is selected and copied in a separate population of children $P\prime$.

**Mutation** Uniform mutation with probability $p_m$ is applied to all children in $P\prime$. With probability $p_m$ all strategies are modified ($\pm$) with $\varepsilon$. If the resulting value is lower than 0, it is set to 0. If it is higher than 1, it is set to 1.

**Termination condition** DES runs either a maximum number of generations which is a parameter of the algorithm and depends on the problem, either until no child can replace a parent for 100 generations. For this the variables *CountReplacements* and *control* in Algorithm 1 are used.

**Algorithm 1** Direct Evolutionary Search algorithm

Randomly generate population;
Evaluate population;
$CountReplacements = 1$;
$control = true$;
**for** $nrgen = 0; ((nrgen < MaxNoGenerations)$ and $(control)); nrgen++$ **do**
  Apply Selection;
  Apply Mutation($p_m$);
  Replace Children;
  $CountReplacements+=$no of children that replace a parent;
  **if** $nrgen = 100$ **then**
    **if** $CountReplacements = 0$ **then**
      $control = false$;
    **else**
      $CountReplacements = 0$;
    **end if**
  **end if**
**end for**
Return Non-Ascended Solutions;
=0

Table 1: Description of the seven games tested

| Name of game | No. of players | No. of strategies | Type of NE |
|---|---|---|---|
| GAME1 | 2 | 2,2 | totally mixed |
| GAME2 | 2 | 2,2 | totally mixed |
| Matching pennies | 2 | 2,2 | totally mixed |
| G1 | 3 | 2,2,2 | totally mixed |
| G2 | 3 | 3,3,3 | mixed |
| O'Neill | 2 | 4,4 | totally mixed |
| Poker | 2 | 4,2 | totally mixed |

Table 2: Descriptive statistics of numerical results obtained by DES using the following parameters: population size 50, $p_m = 0.5$, $\varepsilon = 0.5$.

| | | | |
|---|---|---|---|
| GAME1 | | | |
| Avg. dist: | 0 | St dev: | 0 |
| Avg. no. gen.: | 844.33 | St dev: | 476.57 |
| Avg. eval.: | 447,117.47 | St dev: | 271909.88 |
| GAME3 | | | |
| Avg. dist: | 7.75094E-17 | St dev: | 3.13E-17 |
| Avg. no. gen.: | 501.00 | St dev: | 141.42 |
| Avg. eval.: | 243,736.93 | St dev: | 74416.31 |
| Matching | pennies | | |
| Avg. dist: | 0 | St dev: | 0 |
| Avg. no. gen.: | 497.67 | St dev: | 125.12 |
| Avg. eval.: | 233,854.53 | St dev: | 56641.84 |
| G1 | | | |
| Avg. dist: | 4.04061E-11 | St dev: | 2.25E-17 |
| Avg. no. gen.: | 11,941 | St dev: | 13424.39 |
| Avg. eval.: | 15,541,134 | St dev: | 17317631 |
| G2 | | | |
| Avg. dist: | 4.04061E-11 | St dev: | 6.98E-18 |
| Avg. no. gen.: | 931 | St dev: | 324.70 |
| Avg. eval.: | 1,076,041 | St dev: | 374858.2 |
| Oneill | | | |
| Avg. dist: | 2.79146E-05 | St dev: | 0.0001 |
| Avg. no. gen.: | 8,217.63 | St dev: | 26628.89 |
| Avg. eval.: | 6,556,652.7 | St dev: | 21598601 |
| Poker | | | |
| Avg. dist: | 3.70074E-18 | St dev: | 1.99E-17 |
| Avg. no. gen.: | 794.33 | St dev: | 240.73 |
| Avg. eval.: | 419,827.93 | St dev: | 126713.48 |

## 2 Numerical results

DES was first tested on a set of simple well known 2 or 3-players with up to 4 actions available to them. The games have been chosen from the GAMBIT [3] set of normal form games. The characteristics of the games are presented in Table 1. Table 2 presents descriptive statistics of the results obtained using DES for the seven games: average and standard deviation values for the minimum distance to the NE of the game for the Non Ascended solutions in the final population, number of evaluations and of generations until DES stopped the search.

The second set of games that was used to test if DES was generated by using the GAMUT distribution [5]. The distribution names, characteristics and rates of success of DES are presented in Table 3.

Table 3: GAMUT distributions and results obtained by DES. Rate of success represents percent of runs in which DES detected a NE. Parameters of DES: population size 100, $p_m = 0.1$

| Distribution | No. of players | No. of actions | Rate of success |
|---|---|---|---|
| Bertrand | 2 | 10/player | 100% |
| Ologopoly | 2 | 20/player | 100% |
| | 2 | 50/player | 100% |
| | 2 | 100/player | 100% |
| | 5 | 2/player | 100% |
| | 5 | 4/player | 100% |
| | 5 | 6/player | 100% |
| Bidirectional LEG | 2 | 10/player | 100% |
| Complete Graph | 2 | 20/player | 100% |
| | 2 | 50/player | 100% |
| | 2 | 100/player | 100% |
| | 5 | 2/player | 100% |
| | 5 | 4/player | 100% |
| | 5 | 6/player | 100% |
| Bidirectional LEG | 2 | 10/player | 100% |
| Random Graph | 2 | 20/player | 100% |
| | 2 | 50/player | 100% |
| | 2 | 100/player | 100% |
| | 5 | 2/player | 100% |
| | 5 | 4/player | 100% |
| | 5 | 6/player | 100% |
| Bidirectional LEG | 2 | 10/player | 100% |
| Star Graph | 2 | 20/player | 100% |
| | 2 | 50/player | 100% |
| | 2 | 100/player | 100% |
| | 5 | 2/player | 100% |
| | 5 | 4/player | 100% |
| | 5 | 6/player | 100% |
| Covariance Game | 2 | 10/player | 100% |
| $\rho = 0.9$ | 2 | 20/player | 100% |
| | 2 | 50/player | 100% |
| | 2 | 100/player | 100% |
| | 5 | 2/player | 100% |
| | 5 | 4/player | 100% |
| | 5 | 6/player | 100% |

# Conclusion

The presented results indicate that it it possible for an evolutionary algorithm to directly detect Nash equilibria of normal form games. A population composed of situations of the game is randomly generated; using selection and mutation operators and the Nash ascendancy relation individuals of the populations are guided towards the Nash equilibrium of a normal form game.

Two sets of problems have been chosen to test the method: the first one consists of games presenting a mixed Nash equilibrium. Results indicate that DES is capable of locating mixed equilibria for the selected games. The second group of games is used to test the scalability of the method: the number of actions available to each player, as well as the number of players are increased for five distributions available in the GAMUT package. Results also indicate the potential of the method, inspite of well known scalability issues associated with evolutionary algorithms.

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