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Simulation Experiments for Improving the Consistency Ratio of Reciprocal Matrices

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Abstract: The consistency issue is one of the hot research topics in the analytic hierarchy process (AHP) and analytic network process (ANP). To identify the most inconsistent elements for improving the consistency ratio of a reciprocal pairwise comparison matrix (PCM), a bias matrix can be induced to efficiently identify the most inconsistent elements, which is only based on the original PCM. The goal of this paper is to conduct simulation experiments by randomly generating millions numbers of reciprocal matrices with different orders in order to validate the effectiveness of the induced bias matrix model. The experimental results show that the consistency ratios of most of the random inconsistent matrices can be improved by the induced bias matrix model, few random inconsistent matrices with high orders failed the consistency adjustment.

Keywords: Reciprocal random matrix, Consistency ratio, induced bias matrix, simulation experiment; analytic hierarchy process (AHP)/analytic network process (ANP)

1 Introduction

In the analytic hierarchy process (AHP) and analytic network process (ANP), the pairwise comparison matrix (PCM hereinafter) originated by Thurstone [1] is one of the most important components, which is used to compare two criteria or alternatives with respect to a given criterion , then a matrix $A = (a_{ij})_{n\times n}$ is built to reflect the direct and indirect judgment relations between pairs of criteria or alternatives with respect to a given criteria, where $a_{ij} > 0, a_{ij} = \frac{1}{a_{ji}}$. All PCMs are then used to derive the priority vectors, and the alternatives can be ranked by aggregating the local priority vectors [2–4]. However, the decision made based on the final priority vectors is effective only if the paired comparison matrices pass the consistency test [5]. In practice, it is usually difficult to obtain a matrix that satisfies the perfect consistency condition (i.e. $a_{ij} = a_{ik}a_{kj}$ for $i, j, k = 1, 2, \ldots, n$). Therefore, Saaty [6,7] proved that the maximum eigenvalue λ_{max} of matrix A always satisfies $\lambda_{max} \geq n$ and the equality holds if and only if A is perfectly consistent. Based on this property, Saaty proposed the consistency ratio (CR) to measure the consistency of a matrix, i.e. the consistency of a matrix is acceptable if the CR is less than 0.1. However, this condition sometimes cannot be satisfied with because of the limitations of experiences and expertise, prejudice as well as the complexity nature of the decision problem [8].

To improve the consistency ratio of a matrix, many models and methods have been proposed over the past few decades. For instance, Harker [9] regarded the largest absolute value(s) in matrix $\left\{v_i\omega_j - a_{ji}^2 v_j\omega_i\right\}$ for all i, where $i > j$, as the most inconsistent element(s). Saaty [6] constructed the deviation differences matrix $B = [b_{ij}] = [|a_{ij} - \omega_i/\omega_j|]$ to identify the most inconsistent entry, where ω_i and ω_j are any two subjective priority weights in the $\omega = (\omega_1, \cdots, \omega_n)$. Based on these models, Xu and Wei [10] generated a near consistent matrix $B = (a_{ij}^{\lambda}(\omega_i/\omega_j)^{1-\lambda})_{n\times n}$ to improve the consistency, where λ is a parameter of the auto-adaptive algorithm. Besides, Saaty [7] and Cao et al. [11] introduced Hadamard operator " $\circ$ " to build a perturbation matrix E and a deviation matrix A, in which $E = (a_{ij}) \circ (\omega_j/\omega_i)]$ and $A = [\omega_i/\omega_j] \circ [a_{ij}/(\omega_i/\omega_j)]$, to identify the most inconsistent entry.

There is a common feature in the previously reviewed models, that is, these models are dependent on the priority weights ω_i and ω_j, but there exist more than 20 priority derivation methods [12–14], and the final priority weights obtained from different methods might be different when the matrix is inconsistent. Therefore, Ergu et al. [8] proposed an induced bias matrix to identify the most inconsistent entry in the original inconsistent matrix A. To do so, three major steps containing seven specific steps were proposed and several numerical examples were used to validate the proposed model. In this paper, we attempt to conduct simulation experiments to further validate the effectiveness of the proposed induced bias matrix (IBM) model by generating randomly millions number of the reciprocal positive matrices with different orders. The step 6 and step 7 proposed in Ergu et al. [8] are further quantified and detailed in order to implement automatically modification.

The remaining parts of this paper are organized as follows. The next section briefly describes the induced bias matrix model. The simulation experiments and some algorithms are performed and proposed in Section 3. Section 4 concludes the paper as well as future research directions.

2 The induced bias matrix model

In Ergu et al. [8], the theorem of induced bias matrix and two corollaries were proposed to identify the most inconsistent entries in a PCM and improve the consistency ratio. For the readers' convenience, we first briefly describe the related theorem and corollaries of the IBM model as preliminary of IBM model.

The Theorem 1 says that *"the induced matrix C=AA-nA should be a zero matrix if comparison matrix is perfectly consistent"*. Based on this theorem, if comparison matrix A is approximately consistent, Corollary 1 derived that *"the induced matrix C=AA-nA should be as close as possible to zero matrix"*. However, if the pairwise matrix is inconsistent, Corollary 2 says that *"there must be some inconsistent elements in induced matrix C deviating far away from zero"*. By this corollary, the largest value in matrix C can be used to identify the most inconsistent element in the original matrix A. Some of the identification processes are presented next.

The procedures of the IBM model proposed in Ergu et al. [8] include three major steps, containing seven specific steps (Details are referred to Ergu et al. [8]). The first five steps are easy to be implemented by MATLAB software in practice, i.e. 1) Construct an induced matrix C=AA-nA; 2) Identify the largest absolute value(s) of elements and record the corresponding row and column; 3) Construct the row vector and column vector using the recorded location; 4) Calculate the scalar product f of the vectors; 5) Compute the deviation elements between a_{ij}

and vectors *f*. However, for Steps 6-7, the definitions are not easy to be quantified and it needs the decision makers to determine when we should use *Method of Maximum*,*Method of Minimum*, and *Method for adjusting* a_{ij} to identify the most inconsistent entries. In the following section, we combine these identification methods to perform the simulation experiment by generating randomly reciprocal matrix in order to validate the effectiveness of the induced bias matrix model.

3 Simulation experiments

3.1 Design of simulation experiments

The simulation experiments were performed to confirm the effectiveness of the induced bias matrix model using random inconsistent reciprocal matrices. We generated randomly 10^6 set of reciprocal matrices with orders 3 to 9, and 10^5 set of reciprocal matrices with orders 10-12, i.e. the entries above the main diagonal of a reciprocal matrix is generated randomly from the 17 numbers $(1/9, 1/8, 1/7, \ldots, 1, 2, 3, \ldots, 9)$ in order to satisfy the Saaty's fundamental 9-point scales, the entries below the main diagonal of the PCM is filled automatically with the corresponding reciprocal value. Then calculating the consistency ratio by the formula proposed by Saaty [7], where λ_{max} is the maximum eigenvalue of matrix A, and n is the order of matrix A. If the $CR < 0.1$, discard the generated matrix, if the $CR \geq 0.1$, then applying the IBM model to modify the inconsistent entry and improve the consistency ratio by the six steps and the combined algorithm, as shown in Figure 1. If the consistency ratio of the generated randomly reciprocal pairwise comparison matrix cannot be reduced to be lower than 0.1, then record the corresponding matrix. The specific procedures of this simulation experiment are shown in Figure 1.

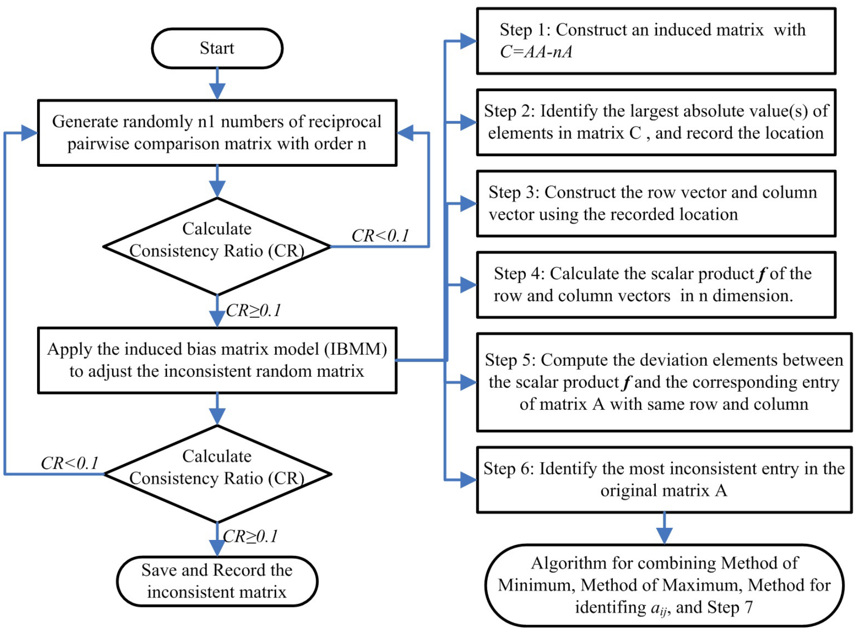

Figure 1: Procedures of simulation experiment by IBM model

For the matrices with $CR \geq 0.1$, the formula of steps 1-6 presented in Ergu et al. [8] are directly used to identify and modify the most inconsistent entries in matrix A as well as improving the consistency ratio. However, the *Method of Maximum, Method of Minimum and Method for* *identifying* a_{ij} proposed in Step 6 and Step 7 involve qualitative observation and judgment, for instance, how many absolute values in vector f can be regarded as more absolute values that are around zero? How to measure the absolute values of a_{ij}, a_{ik} and a_{kj} are too large or too small by quantifying terms? Therefore, the following Algorithm is used to combine the previously mentioned identification processes.

3.2 Algorithms of simulation experiments

In order to simulate the induced bias matrix model, the program codes with two input parameters, n and n1 were written by Matlab 7.0 to randomly generate reciprocal matrices, in which n denotes the numbers of random reciprocal matrix, while n1 represents the number of simulation. For the space limitations, we omitted the first five steps, and the following Algorithm is used to combine the *Method of Maximum*,*Method of Minimum*, and *Method for adjusting* a_{ij}.

Algorithm 1: Improving the consistency ratios of the random reciprocal matrices with $CR \geq 0.1$
Input: Matrix Order, n; Number of simulation, n1
Output: Matrices with $CR \geq 0.1$
Process:
01. C=AA-nA % Matrix A is the generated randomly reciprocal matrix with $CR \geq 0.1$
02. If $c_{ij} < 0$
03. Adjust a_{ij} using $a_{ij} = \frac{1}{n-2}\sum_{k=1}^{n} a_{ik}a_{kj}$
04. End % Method for identifying $a_{ij}(1)$
05. If $c_{ij} > 0$ & & $min(f) == 0$ % We can obtain that a_{ij} is inconsistent whether it is too large or too small, in which f is the vector product
06. Adjust a_{ij} using $\widetilde{a}_{ij} = \frac{1}{n-2}\sum_{k=1}^{n} a_{ik}a_{kj}$
07. End % Method for identifying $a_{ij}(2)$
08. If $c_{ij} > 0$ % a_{ij} and a_{ik} (or a_{kj}) might have problematic
09. $[m, k] = max(f)$; % m is the element with the largest value in vector f, while k is the corresponding location.
10. If $c_{ik} < 0$ & & $c_{kj} \geq 0$ % a_{ik} is problematic (too large).
11. Adjust c_{ik} using $a_{ik} + c_{ik}/(n-2)$.
12. Break
13. End
14. If $c_{ik} \geq 0$& & $c_{kj} < 0$ % a_{kj} is problematic and large
15. Adjust a_{kj} using $a_{kj} + c_{kj}/(n-2)$
16. Break
17. End
19. If $c_{ik} < 0$ & & $c_{kj} < 0$ % c_{ik} and c_{kj} are problematic
20. If $abs(c_{ik}) >= abs(c_{kj})$
21. Adjust a_{ik} using $a_{ik} + c_{ik}/(n-2)$
22. Break
23. Else
24. Adjust a_{kj} using $a_{kj} + c_{kj}/(n-2)$
25. Break
26. End
27. End

28. If $f(k) > 0$ & & $c_{ik} >= 0$ & & $c_{kj} >= 0$ % It is unreasonable to occur simultaneously, if it does occur, then go to adjust the second largest value.
29. $c_{ij} = 0$;
30. End
31. End
32. Calculate the CR; % see Algorithm 2
33. If $CR < 0.1$
34. Break
35. End.

Algorithm 2: Calculating the consistency ratio of the modified matrix B
Input: Modified random matrix B
Output: Consistency ratio CR.
Process:

```
01. n=length(B);        % B is reciprocal matrix
02. RI=[1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 0 0 0.52 0.89 1.12 1.26 1.36 1.41 1.46 1.49 1.52 1.54
1.56 1.58 1.59];
03. [a,b]=eig(B);
04. [c1,d1]=max(b);        % c1 is maximum value in each column, d1 is the corresponding row
of each element
05. [e1,f1]=max(c1);        % e1 is the largest element, f1 is the corresponding column.
06. CI=(e1-n)/(n-1);
07. CR=CI/RI(2,n);
```

3.3 Experimental results

In this section, we do not attempt to optimize the program codes for speed, therefore, we set the matrix order n to be 3-12, and the simulation number n1=10^6 for the matrices with orders 3-9. For the matrix with orders 9-12, we only simulated 10^5 numbers of randomly reciprocal matrices. The results of simulation experiments are shown in Table 1. It can be seen that some of the random reciprocal matrices with orders from 3 to 6 passed the consistency test, for instance, 206130 random matrices with order 3 passed the consistency test among 10^6 matrices, while 73 random matrices with order 6 passed the consistency text. However, all random matrices with orders 7-12 did not pass the consistency test. For the random matrices with $CR \geq 0.1$, the proposed IBM model was used to modify the most consistent entries and improve the consistency ratio. Table 1 shows that the consistency ratios of all the inconsistent random matrices with $CR \geq 0.1$ and orders 3-7 have been improved and lower than 0.1 after the proposed IBM model is used to modify the random matrices, as shown in Figures 2-7, while some matrices still failed the consistency test, the numbers are 3 for order 8, 5 for order 9, 1 for order 10, 2 for order 11 and 13 for order 12, as shown in Figures 8-12. The corresponding simulation Figures for 10^6 random matrices with orders 3 to 9, and 10^5 random matrices with orders 10-12 are shown in Figures 2-10.

Table 1 Simulation experiments for randomly generated matrices with different orders

Matrix Order	Number of simulation	Number of matrices with $CR \geq 0.1$	Failed matrices	Succeeded Matrices
3	1000000	793870	0	793870
4	1000000	968083	0	968083
5	1000000	997518	0	997518
6	1000000	999927	0	999927
7	1000000	1000000	0	100000
8	1000000	1000000	3	999997
9	1000000	1000000	5	999995
10	100000	100000	1	99999
11	100000	100000	2	99998
12	100000	100000	13	99987

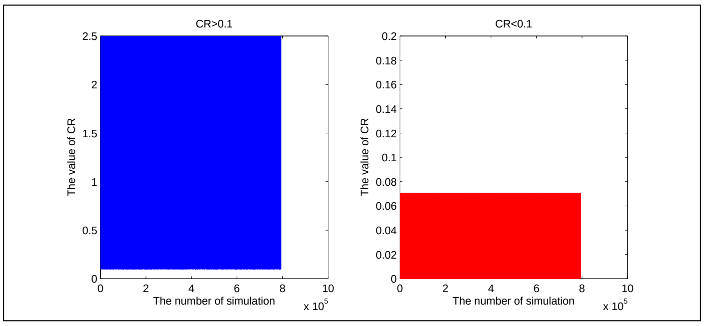

Figure 2: Simulation experiment for 10^6 randomly generated matrices with order 3

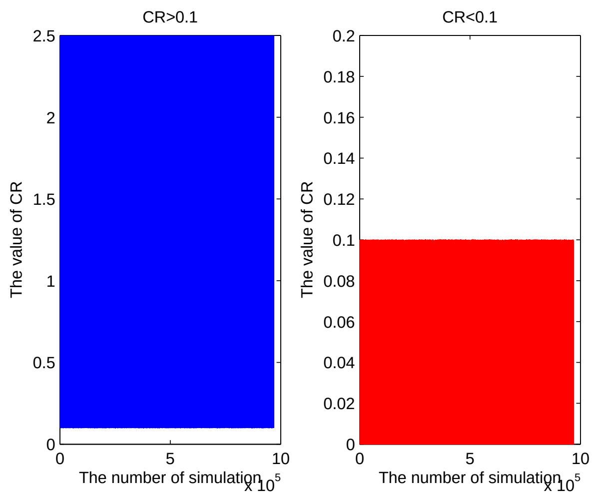

Figure 3: Simulation experiment for 10^6 randomly generated matrices with order 4

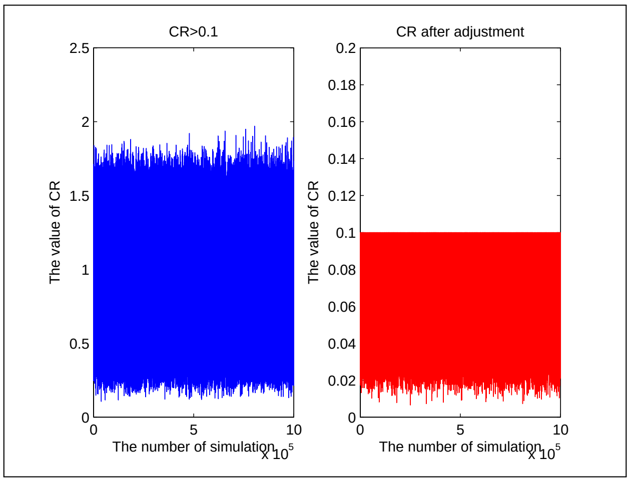

Figure 4: Simulation experiment for 10^6 randomly generated matrices with order 5

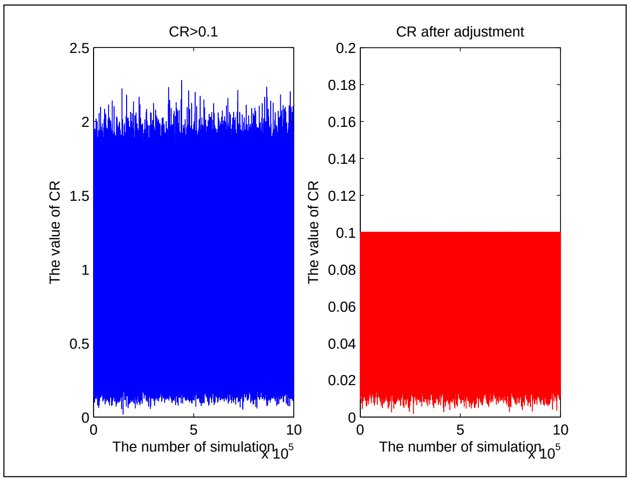

Figure 5: Simulation experiment for 10^6 randomly generated matrices with order 6

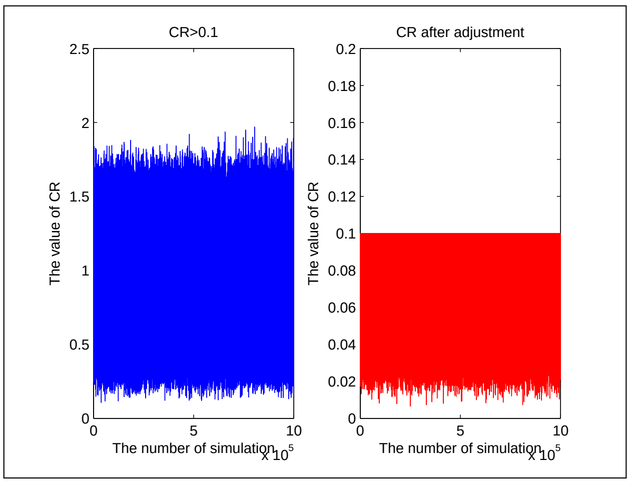

Figure 6: Simulation experiment for 10^6 randomly generated matrices with order 7

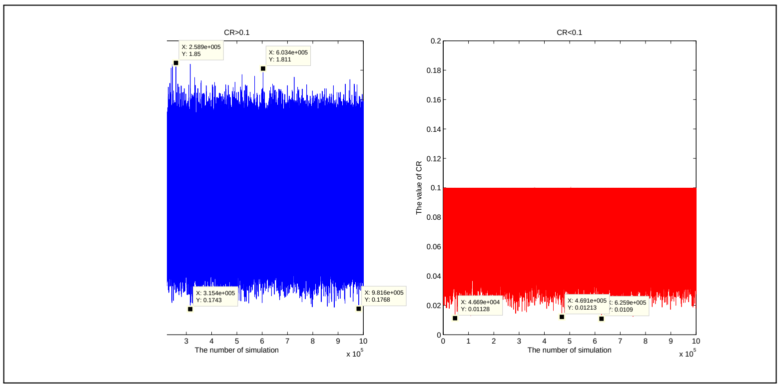

Figure 7: Simulation experiment for 10^6 randomly generated matrices with order 8

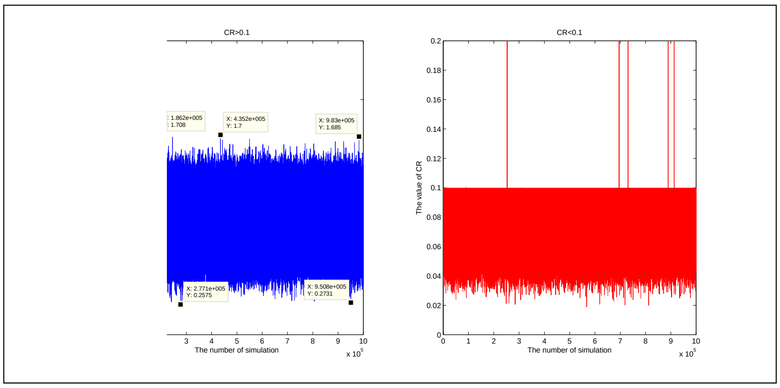

Figure 8: Simulation experiment for 10^6 randomly generated matrices with order 9

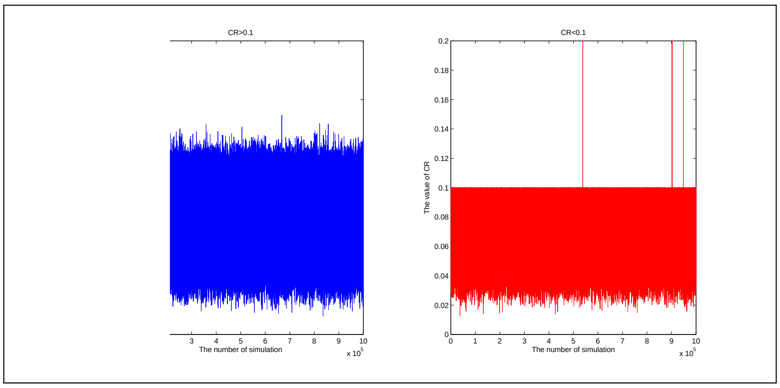

Figure 9: Simulation experiment for 10^5 randomly generated matrices with order 10

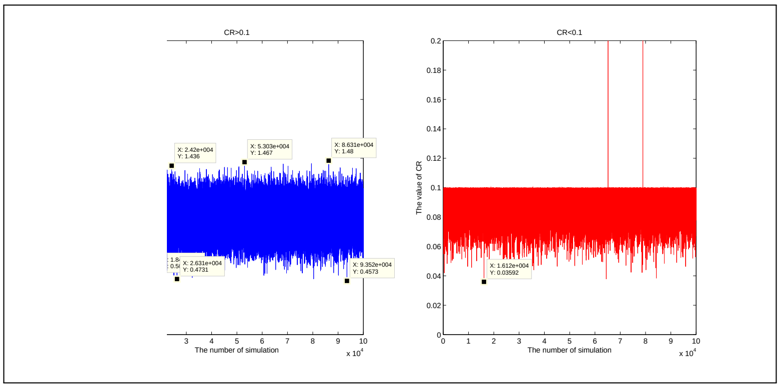

Figure 10: Simulation experiment for 10^5 randomly generated matrices with order 11

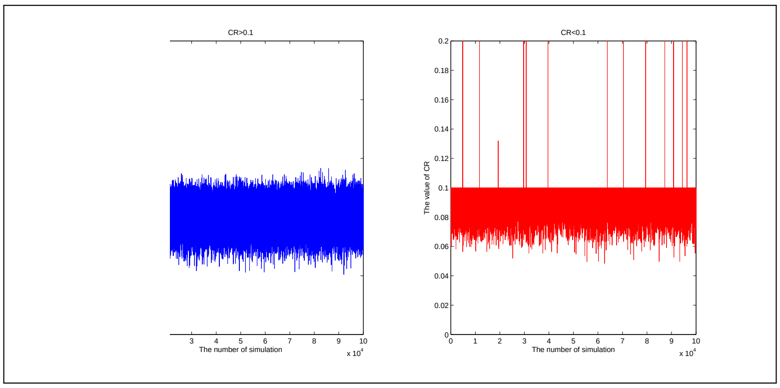

Figure 11: Simulation experiment for 10^5 randomly generated matrices with order 12

4 Conclusions

In this paper, some of the identification processes proposed in Ergu et al. [8] were combined to implement the programming. Based on these combinations, an algorithm was proposed and simulation experiments on random reciprocal matrices with different orders were conducted to validate the effectiveness of the induced bias matrix model. We found that some matrices generated randomly could pass the consistency test, and the higher the orders of matrices are, the less the matrices have $CR < 0.1$. When the orders of random matrices increase to 7, all matrices generated randomly have $CR \geq 0.1$, and they need to be adjusted. After the algorithm of the induced bias matrix (IBM) model was applied to these matrices, all the consistency ratios of random matrices with orders 3-7 were improved and less than the threshold 0.1, while fewer matrices with order higher than 8 still could not be modified satisfactorily. However, we believe that the consistency of the pairwise comparison matrices provided by experts will be much better than the consistency of random matrices, thus the proposed IBM model is capable of dealing with the consistency of a PCM.

Although the results of the simulation experiments show the effectiveness of the IBM model, the experimental findings also reveal the failed tendency will increase with the increase of the matrices order. The failed matrices should be paid more attention to and analyze the reason why it failed the consistency test, we leave it for further research in future.

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